Day 13, R 2/22/2024

Topic 6: Operators, inhomogeneous DES, ERF, SRF (day 1 of 2)

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1 Agenda

- Finish Topic 5 problems
- Operators, the operator $D = \frac{d}{dt}$
- Polynomial operators $P(D)$
- Linearity of $P(D)$ and the superposition principle
- Substitution rule: $P(D)e^{rt} = P(r)e^{rt}$
- Tomorrow: ERF, SRF

2 Operators

Same thing – new symbols and terminology.

$D = \dfrac{d}{dt}$ is called a differential operator.

An operator acts on a function to produce another function.

Example 1. $D(t^2 + t^3) = \frac{d}{dt}(t^2 + t^3) = 2t + 3t^2, \qquad D(e^{at}) = ae^{at}, \qquad Df = f'$.

Likewise, $D^2 = \dfrac{d^2}{dt^2}, \quad D^3 = \dfrac{d^3}{dt^3}$.

Example 2. $D^2 e^{at} = a^2 e^{at}, \qquad D^3 e^{at} = a^3 e^{at}, \quad \ldots$

Identity operator: I: $I(f) = f$.

General operator: T: $Tf =$ "T applied to f".

3 Polynomial operators: $P(D)$

Example 3. Suppose $P(r) = r^2 + 8r + 7 \longrightarrow P(D) = D^2 + 8D + 7I$.

So, $P(D)x = D^2x + 8Dx + 7Ix = x'' + 8x' + 7x$.

3.1 Comparing old and new notation

	old	new
DE	$x'' + 8x' + 7x = 0$	$P(D)x = 0$
Characteristic eq.	$r^2 + 8r + 7 = 0$	$P(r) = 0$

Here, $P(D) = D^2 + 8D + 7I, \quad P(r) = r^2 + 8r + 7$

4 Linearity and the superposition principle

Suppose x_1, x_2 are functions, c_1, c_2 are constants. Then

$$D(c_1x_1 + c_2x_2) = (c_1x_1 + c_2x_2)' = c_1x_1' + c_2x_2' = c_1Dx_1 + c_2Dx_2.$$

Likewise, $D^2(c_1x_1 + c_2x_2) = c_1D^2x_1 + c_2D^2x_2$. This is the heart of all of our superposition/linearity principles. Operators with this property are called linear operators. We write out the definition:

Definition: T is a linear operator if

$$T(c_1x_1 + c_2x_2) = c_1Tx_1 + c_2Tx_2.$$

So, D, D^2, D^3, $D^2 + 3D$, ..., $P(D)$ are all linear.

4.1 Superposition principle (equivalent to linearity)

If x_1 solves $P(D)x = f_1$, i.e., $P(D)x_1 = f_1$

and x_2 solves $P(D)x = f_2$, i.e., $P(D)x_2 = f_2$

then $x = c_1x_1 + c_2x_2$ solves $P(D)x = c_1f_1 + c_2f_2$ (c_1, c_2 constants).

Proof. Use the linearity of $P(D)$.

$$P(D)x \underset{\uparrow}{=} P(D)(c_1x_1 + c_2x_2) \underset{\uparrow}{=} c_1P(D)x_1 + c_2P(D)x_2 \underset{\uparrow}{=} c_1f_1 + c_2f_2 \quad \blacksquare$$

(definition of x) (linearity of $P(D)$) (assumption $P(D)x_1 = f_1$ and $P(D)x_2 = f_2$)

4.2 Variations of the superposition principle

Homogeneous equations:

$$\text{If } P(D)x_1 = 0 \text{ and } P(D)x_2 = 0, \text{ then } P(D)(c_1x_1 + c_2x_2) = 0.$$

Proof. Linearity of $P(D)$.

Inhomogeneous equations:

$$\text{If } P(D)x_p = f \text{ and } P(D)x_h = 0, \text{ then } P(D)(x_p + x_h) = f.$$

Proof. Linearity of $P(D)$.

5 Substitution rule

Substitution rule: $P(D)e^{at} = P(a)e^{at}$.

Example 4. Say $P(D) = D^2 + 8D + 7I$. Then

$$\begin{aligned} P(D)e^{at} &= D^2e^{at} + 8De^{at} + 7Ie^{at} \\ &= a^2e^{at} + 8ae^{at} + 7e^{at} \\ &= (a^2 + 8a + 7)e^{at} \\ &= P(a)e^{at} \quad \blacksquare \end{aligned}$$