Solutions Day 13, R 2/22/2024

Topic 6: Operators, inhomogeneous DEs, ERF, SRF

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Problem 1. *Let $P(r) = r^2 + 8r + 7$.*

(a) *Compute $P(D)t^3$, i.e., "$P(D)$ applied to t^3".*

Solution: $P(D) = D^2 + 8D + 7I$. So, $P(D)x = x'' + 8x' + 7x \quad \Rightarrow P(D)t^3 = 6t + 24t^2 + 7t^3$.

(b) *Compute $P(D)e^{rt}$.*

Solution: $P(D)e^{rt} = r^2e^{rt} + 8re^{rt} + 7e^{rt} = (r^2 + 8r + 7)e^{rt} = P(r)e^{rt}$.

(c) *Write $P(D)x = 0$ out the long way. What is the characteristic equation?*

Solution: $P(D)x = 0 \quad \Leftrightarrow x'' + 8x' + 7x = 0$.

Characteristic equation: $\quad r^2 + 8r + 7 = 0 \quad$ or $\quad P(r) = 0$.

Problem 2.

(a) *Show that $D^2 + 5D = \frac{d^2}{dt^2} + 5\frac{d}{dt}$ is a linear operator.*

Solution: We need to show that $(D^2+5D)(c_1x_1+c_2x_2) = c_1(D^2+5D)x_1+c_2(D^2+5D)x_2$, where c_1, c_2 are scalars. This is just a bit of algebra:

$$\begin{aligned}(D^2 + 5D)(c_1x_1 + c_2x_2) &= (c_1x_1 + c_2x_2)'' + 5(c_1x_1 + c_2x_2)' \\ &= c_1(x_1'' + 5x_1') + c_2(x_2'' + 5x_2') \\ &= c_1(D^2 + 5D)x_1 + c_2(D^2 + 5D)x_2 \qquad \blacksquare\end{aligned}$$

(b) *Show that T, defined by $Tf = f^2$ is not linear.*

Solution: This is like Part (a) except we show that equality doesn't hold.

$$T(x_1 + x_2) = (x_1 + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2 = Tx_1 + Tx_2 + 2x_1x_2 \neq Tx_1 + Tx_2.$$

Since $T(x_1 + x_2) \neq Tx_1 + Tx_2$, T is not a linear operator.

Problem 3. *Solve $x'' + 8x' + 7x = e^{2t}$ by guessing a solution of the form $x = ce^{2t}$.*

Solution: We try $x = ce^{2t}$, i.e., plug this into the DE.

$$x''+8x'+7x = e^{2t} \quad \Rightarrow 4ce^{2t}+16ce^{2t}+7ce^{2t} = e^{2t} \quad \Rightarrow c(4+16+7)e^{2t} = e^{2t} \quad \Rightarrow c = \frac{1}{27}.$$

Thus, $x(t) = \frac{1}{27}e^{2t}$ is a solution.

Problem 4.

(a) *Use complex replacement to compute $D^3(e^t \cos t)$.*

Solution: Let $x = D^3(e^t \cos t)$. Complex replacement says to replace $e^t \cos t$ by $e^te^{it} = e^{(1+i)t}$. (Careful explanation of this below.)

Then, if $z = D^3(e^{(1+i)t})$, we have $x = \text{Re}(z)$. Computing:

$$z = D^3(e^{(1+i)t}) = (1 + i)^3e^{(1+i)t}.$$

Since $1+i=\sqrt{2}e^{i\pi/4}$, $(1+i)^3 = 2^{3/2}e^{i3\pi/4}$. We now have,

$$z = 2^{3/2}e^{i3\pi/4}e^{(1+i)t} = 2^{3/2}e^t e^{i(t+3\pi/4)}.$$

Taking the real part

$$x = \mathrm{Re}(z) = \mathrm{Re}\left(2^{3/2}e^t e^{i(t+3\pi/4)}\right) = \mathrm{Re}\left(2^{3/2}e^t\left[\cos\left(t+\frac{3\pi}{4}\right)+i\sin\left(t+\frac{3\pi}{4}\right)\right]\right) = \boxed{2^{3/2}e^t\cos\left(t+\frac{3\pi}{4}\right)}.$$

Careful justification of complex replacement:

$$\begin{array}{lll} \text{We have} & x & = D^3(e^t\cos t). \\ \text{Let} & y & = D^3(e^t\sin t). \\ \text{and let} & z & = x+iy = D^3(e^t\cos t)+iD^3(e^t\sin t) \\ & & = D^3(e^t\cos t + ie^t\sin t) \quad (\text{linearity of } D^3) \\ & & = D^3(e^t e^{it}) \qquad (\text{Euler's formula}) \\ & & = D^3(e^{(1+i)t}) \end{array}$$

So, $z = D^3(e^{(1+i)t}$, and, by definition of z, $x = \mathrm{Re}(z)$.

(b) *Make use of your work in Part (a) to compute $D^3(e^t\sin t)$.*

Solution: The complexification in Part (a), showed $D^3(e^t\sin t) = \mathrm{Im}(z) = 2^{3/2}e^t\sin\left(t+\frac{3\pi}{4}\right)$.