Day 14, F 2/23/2024

Topic 6: P(D), ERF, SRF (day 2 of 2)
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1 Agenda

- Review
- Exponential response formula (ERF)
- Sinusoidal response formula (SRF)
- Driven damped harmonic oscillators (in problems)

2 Review

Operator D: $D = \frac{d}{dt}$, $P(r) = r^2 + 8r + 7 \longleftrightarrow P(D) = D^2 + 8D + 7I$.

Linearity: Operator T is linear if

$$T(c_1 f + c_2 g) = c_1 Tf + c_2 Tg \text{ for constants } c_1,\, c_2.$$

Important: D, D^2, $P(D)$ are all linear

Do Problem 2 from yesterday (if not already done).

Superposition principles

General:	If $P(D)x_1 = f_1$, $P(D)x_2 = f_2$,	then	$P(D)(c_1x_1 + c_2x_2) = c_1f_1 + c_2f_2$.
Homogeneous:	If $P(D)x_1 = 0$, $P(D)x_2 = 0$,	then	$P(D)(c_1x_1 + c_2x_2) = 0$.
Particular + homogeneous:	If $P(D)x_p = f$, $P(D)x_h = 0$,	then	$P(D)(x_p + x_h) = f$.
Observation:	If $P(D)x_1 = f$, $P(D)x_2 = f$,	then	$P(D)(x_1 - x_2) = 0$.

Proof of all of these: Linearity!

Substitution rule: $P(D)e^{at} = P(a)e^{at}$.

3 Exponential response formula (ERF)

Consider the inhomogeneous DE with exponential input: $P(D)x = e^{at}$,

e.g., $(D^2 + 8D + 7I)x = e^{2t} \quad \longleftrightarrow \quad x'' + 8x' + 7x = e^{2t}$.

Exponential response formula (ERF) for the equation $P(D)x = e^{at}$.

If $P(a) \neq 0$, this has a particular solution $x_p(t) = \dfrac{e^{at}}{P(a)}$ (ERF)

If $P(a) = 0$, $P'(a) \neq 0$, then $x_p(t) = \dfrac{te^{at}}{P'(a)}$ (Extended ERF)

If $P(a) = 0$, $P'(a) = 0$, $P''(a) \neq 0$, then $x_p(t) = \dfrac{t^2e^{at}}{P''(a)}$ (Extended ERF)

Example 1. Find a particular solution (any one) to $x'' + 8x' + 7x = e^{2t}$.

Solution: $P(r) = r^2 + 8r + 7$, so $P(2) = 27$. Therefore, by the ERF, $x_p(t) = \dfrac{e^{2t}}{27}$ is a solution.

Reason: Method of optimism: Guess solution $x(t) = ce^{2t}$.
Algebra will determine which, if any, values of c work: substitute $x(t) = ce^{2t}$ into the DE:

$$\underbrace{x'' + 8x' + 7x}_{\text{left side of DE}} = P(D)x = P(D)\underbrace{(ce^{2t})}_{x(t)} \overbrace{=}^{\text{substitution rule}} cP(2)e^{2t} = \underbrace{e^{2t}}_{\text{right side of DE}}$$

So we need $cP(2) = 1$, which gives $c = \dfrac{1}{P(2)}$, thus $x(t) = \dfrac{e^{2t}}{P(2)} = \dfrac{e^{2t}}{27}$ (This is the ERF!)

Do some problems.

4 Sinusoidal response formula (SRF)

Consider the DE with sinusoidal input: $P(D)x = \cos(\omega t)$,
e.g., $(D^2 + 8D + 7I)x = \cos(2t) \quad \longleftrightarrow \quad x'' + 8x' + 7x = \cos(2t)$.

Sinusoidal response formula (SRF) for the equation $P(D)x = \cos(\omega t)$.
If $P(i\omega) \neq 0$, this has a particular solution

$$x_p(t) = \frac{\cos(\omega t - \phi)}{|P(i\omega)|}, \quad \text{where } \phi = \text{Arg}(P(i\omega)). \qquad \text{(SRF)}$$

If $P(i\omega) = 0$, $P'(i\omega) \neq 0$ this has a particular solution

$$x_p(t) = \frac{t\cos(\omega t - \phi)}{|P'(i\omega)|}, \quad \text{where } \phi = \text{Arg}(P'(i\omega)). \qquad \text{(extended SRF)}$$

The pattern continues if $P(i\omega) = 0$, $P'(i\omega) = 0$, $P''(i\omega) \neq 0$ etc.

Example 2. Solve $x'' + 8x' + 7x = \cos(2t)$.
Solution: $P(r) = r^2 + 8r + 7$. So, $P(2i) = -4 + 16i + 7 = 3 + 16i$.

Polar form: $|P(2i)| = \sqrt{265}$, $\boxed{\phi = \text{Arg}(P(2i)) = \tan^{-1}\left(\frac{16}{3}\right) \text{ in Q1}}$.

So, by the SRF, $\boxed{x_p(t) = \dfrac{\cos(\omega t - \phi)}{\sqrt{265}}}$ is a solution.

Reason: (Probably won't do in class. It's in the Topic 6 notes)

Use complex replacement and the ERF:

Want to solve $P(D)x = \cos(\omega t)$.

Let y be a solution to $P(D)y = \sin(wt)$.

By linearity $P(D)\underbrace{(x+iy)}_{z} = \underbrace{\cos(\omega t) + i\sin(\omega t)}_{e^{i\omega t}}$

That is, $P(D)z = e^{i\omega t}$ and $x = \text{Re}(z)$: By the ERF, $z_p(t) = \dfrac{e^{i\omega t}}{P(i\omega)}$.

Polar form: $P(i\omega) = |P(i\omega)|\, e^{i\phi}$, where $\phi = \text{Arg}(P(i\omega))$.

So, $z_p(t) = \dfrac{e^{i\omega t}}{|P(i\omega)|e^{i\phi}} = \dfrac{1}{|P(i\omega)|} e^{i(\omega t - \phi)}$.

Finally, $x_p(t) = \text{Re}(z_p(t)) = \dfrac{1}{|P(i\omega)|}\cos(\omega t - \phi)$ ■