Day 16, T 2/27/2024

Topic 8: Stability
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1 Agenda

- Time invariance
- Stability
  –mathematical
  –physical
- Routh-Hurwitz criteria for stability
- Maxwell and exploding steam engines

2 Time invariance

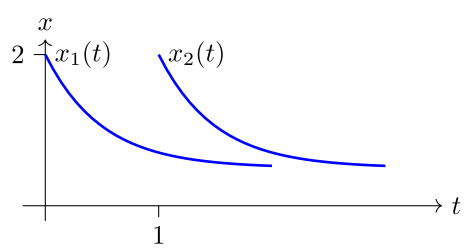
Time invariance: The same initial state and input produces the same output, no matter what time the system starts.

Constant coefficient systems are time invariant.

Example 1. Consider the two systems (same input and IC, but starting at different times)

$$x_1' + 2x_1 = 1, \quad x(0) = 2 \longrightarrow x_1(t) = \frac{1}{2} + \frac{3}{2}e^{-2t}$$
$$x_2' + 2x_2 = 1, \quad x(1) = 2 \longrightarrow x_2(t) = \frac{1}{2} + \frac{3}{2}e^{-2(t-1)}.$$

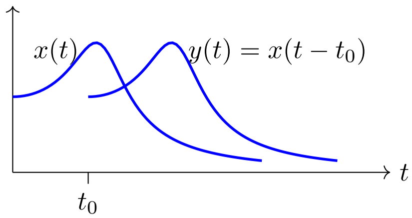
So, $x_2(t) = x_1(t-1)$, i.e., the graph of x_2 is a shifted copy of the graph of x_1.



Example 2. If
$$x'' + 8x' + 7x = f(t), \quad x(0) = 1,\ x'(0) = 2,$$
then $y(t) = x(t - t_0)$ satisfies
$$y'' + 8y' + 7y = f(t - t_0), \quad y(t_0) = 1,\ y'(t_0) = 2.$$

(Same input and IC, both shifted in time.)

This is easy to check: $y''(t) + 8y'(t) + 7y(t) = x''(t - t_0) + 8x'(t - t_0) + 7x(t - t_0) = f(t - t_0)$.



3 Stability

3.1 Mathematical stability

By mathematic stability we mean: Initial conditions don't affect long-term behavior.

For constant coefficient systems $P(D)x = f$:

stability = all homogeneous solutions decay to 0 as t increases
= all characteristic roots have negative real part
= pole diagram has all roots in the left half-plane

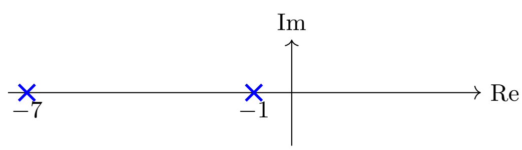
Example 3. Consider $x'' + 8x' + 7x = \cos(2t)$, $x(0) = b_0$, $x'(0) = b_1$. Show this satisfies each of the descriptions of stability.

Solution: The characteristic roots are -1, -7.

• All homogoneous solutions decay to 0: The general homogeneous solution is $x_h(t) = c_1 e^{-t} + c_2 e^{-7t}$. Because of the negative exponents, these all decay to 0 as t gets large.

• All roots have negative real part: This is clear.

• Pole diagram has all roots in the left half-plane:



• Long-term behavior is not affected by the initial conditions:

This takes some algebra. The general solution to the DE is

$$x(t) = \frac{\cos(2t - \phi)}{\sqrt{265}} + c_1 e^{-t} + c_2 e^{-7t}.$$

So the initial conditions give

$$x(0) = \frac{\cos(-\phi)}{\sqrt{265}} + c_1 + c_2 = b_0$$

$$x'(0) = \frac{-2\sin(-\phi)}{\sqrt{265}} - c_1 - 7c_2 = b_1$$

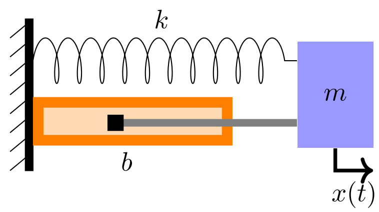
We can solve these equations for c_1, c_2. Whatever the values of c_1 and c_2, we see that $x(t)$ goes asymptotically to $\frac{\cos(2t-\phi)}{\sqrt{265}}$. That is, the initial conditions $x(0) = b_0$, $x'(0) = b_1$ do not affect the long-term behavior of the system.

Important note: Stability is about the system, not the input.

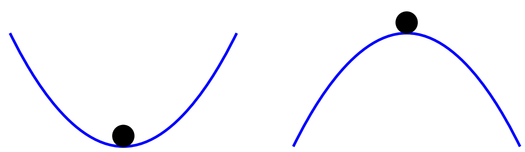
3.2 Physical stability

A system with a single equilibrium is called stable if, when there is no external input, the system always returns to that equilibrium.

Example 4. Damped harmonic oscillators are stable



Example 5. A ball in a cup will settle to the botton, So this is a stable equilibrium. A ball on top of a hill will roll away from the unstable equilbrium at the top.



Stable and unstable equilibria

3.3 Connection between mathematical and physical stability

$P(D)x = f$ is mathematically stable means $x_h(t) \longrightarrow 0$ as $t \longrightarrow 0$.

Physical stablity means the unforced system $P(D)x = 0$ always returns to the equilibrium at $x = 0$, i.e., $x_h(t)$ goes to 0.

4 Routh-Hurwitz criteria for stability in $P(D)x = f$

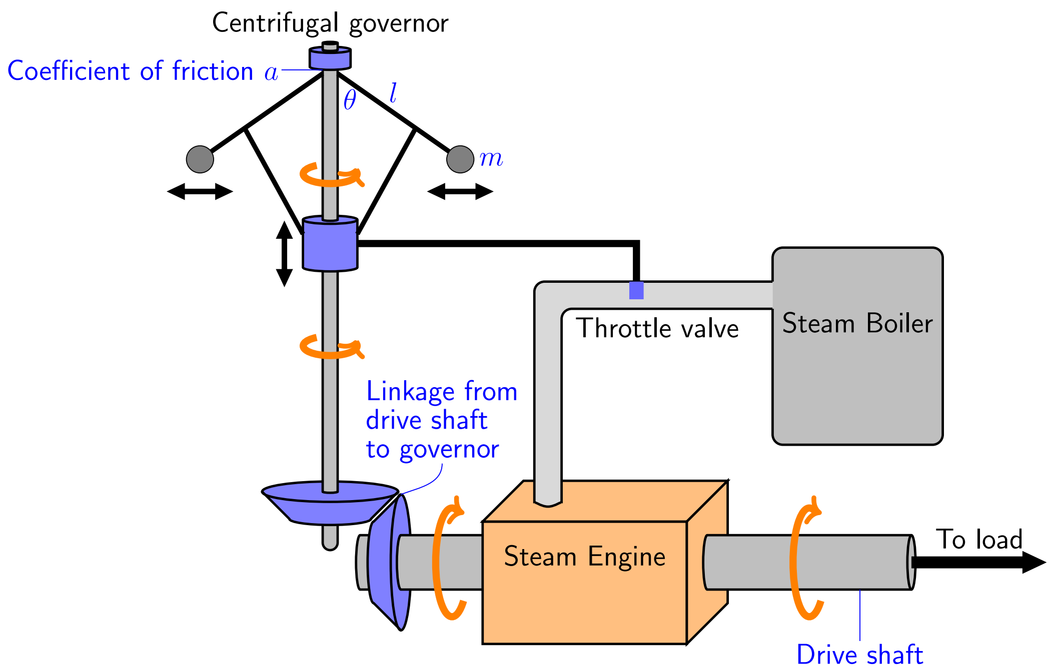
The key point is that we can determine stability directly from the coefficients of $P(D)$ without having to compute roots.

1^{st}-order	$x' + kx = f(t)$	stable $\longleftrightarrow$ $k > 0$
2^{nd}-order	$mx'' + bx' + kx = f(t)$, $m > 0$	stable $\longleftrightarrow$ $b, k > 0$
3^{rd}-order	$x''' + ax'' + bx' + cx = f(t)$	stable $\longleftrightarrow$ $a, b, c > 0$ **and** $ab > c$

You should know 1^{st} and 2^{nd}-order. You should know there is a condition for 3^{rd}-order and higher, but we won't expect you to memorize it.

5 Maxwell and exploding steam engines*

In his paper *On Governors***, J. C. Maxwell explored the dynamics and stability of governors. This was a foundational paper in control theory, which introduced the notion that all the roots of the characteristic equation having negative real part was necessary for stability.



If the drive shaft speeds up, the governor turns faster, so the balls rise. This closes the throttle valve, which limits the steam to the engine, so it slows down.

There is an equilibrium position θ_0 for the angle θ. The linearization*** of the nonlinear model for θ near the equilbrium is

$$\theta''' + \frac{a}{m}\theta'' + b\theta' + c\theta = 0 \text{ (Maxwell)}$$

Here, a is the coefficient of friction in the governor's bearings and b, c are physical constants determined by the configuration and load.

By Routh-Hurwitz (discovered by Maxwell), this is unstable if $\frac{a}{m} \cdot b < c$.

If the machinists make the bearings nearly frictionless, i.e., make a small, then $\frac{a}{m} \cdot b < c$ and the system is unstable. The governor in the unstable system is too reactive and can cause the speed of the engine to oscillate with increasing amplitude, leading to catastrophic failure.

*Much of this section is taken from an unpublished writeup by Nirav Shah, called "Derivation of the Maxwell/Vyshnegradskii Stability Criterion for the Watts Centrifugal Governor"

**Maxwell J.C., "On Governors", *Proc. of the Royal Society*, vol 16, Mar. 1868

***We'll learn about linearization in Topic 28.