Day 18, R 2/29/2024

**Topic 9: Engineering language: input, gain, phase lag (day 1 of 3)**

Jeremy Orloff

# 1 Agenda

- Input
- Naming the parts of solutions
- Tomorrow: gain and resonance

# 2 Setup: stable systems

We'll assume all our systems are stable.

– Since $x_h(t) \longrightarrow 0$, we will focus on particular solutions.

– We will only consider sinusoidal input: $B\cos(\omega t)$.

# 3 Input

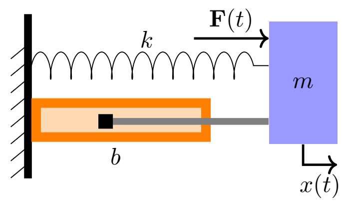
Input is an engineering term. It is up to the engineer to tell us what we should call the input.

**Examples:**

1. External force on the mass in a spring-mass-damper system.

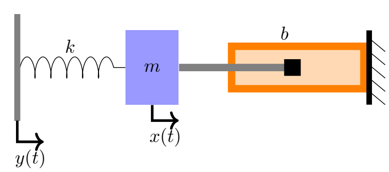
Model: $mx'' + bx' + kx = F(t)$.

Input: It's natural to call the force $F(t)$ the input.



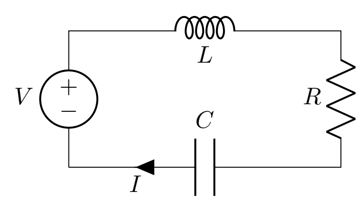

2. Drive the mass by pushing on the end of the spring. $y(t)$ is the position of the end of the spring.

Model: $mx'' + bx' + kx = ky$. Natural to call $y(t)$ the input.




3. Voltage $V$ driving a circuit with current $I$.

Model: $LI'' + RI' + \frac{1}{C}I = V'$. Call $V$ the input.



# 4 Naming the parts of solutions

**Example 1.** Consider the system $2x'' + 5x' + 3x = 3B\cos(\omega t)$, where the engineer has told us that $B\cos(\omega t)$ is considered the input.

Solve the DE for the general real-valued solution.

Name the pieces of the solution.

**Solution:** $P(r) = 2r^2 + 5r + 3$. So, $P(i\omega) = 3 - 2\omega^2 + 5i\omega$.

Sinusoidal solution: (SRF) $x_p(t) = \dfrac{3B\cos(\omega t - \phi(\omega))}{|P(i\omega)|}$, $\phi(\omega) = \text{Arg}(P(i\omega))$. (We won't bother computing $\phi(\omega)$ and $|P(i\omega)|$.)

Homogeneous solution: Roots $-3/2$, $-1$ $\longrightarrow$ $x_h(t) = c_1 e^{-\frac{3}{2}t} + c_2 e^{-t}$.

General solution: $$x(t) = \underbrace{\frac{3B\cos(\omega t - \phi(\omega))}{|P(i\omega)|}}_{\text{periodic or sinusoidal response}} + \underbrace{c_1 e^{-\frac{3}{2}t} + c_2 e^{-t}}_{\text{transient (goes to 0)}}$$

Input features:

| | | |
|---|---|---|
| $B\cos(\omega t)$ | = | input (given to us) |
| $B$ | = | input amplitude |
| $\omega$ | = | input frequency (angular frequency in radians/time) |

Solution (output/response) features:

$$\frac{3B\cos(\omega t - \phi(\omega))}{|P(i\omega)|} \quad = \quad \text{output (depends on } \omega.)$$

$$\frac{3B}{|P(i\omega)|} \quad = \quad \text{output amplitude (depends on } \omega.)$$

$$\phi(\omega) \quad = \quad \text{phase lag in radians (depends on } \omega.)$$

$$g(\omega) = \frac{\text{output amplitude}}{\text{input amplitude}} = \frac{3}{|P(i\omega)|} \quad = \quad \text{gain (depends on } \omega.)$$

(If output and input have the same dimension, then $g$ is dimensionless,

e.g., if input and output are both voltages, then $g$ is called the voltage gain.)

$$\frac{\phi(\omega)}{\omega} \quad = \quad \text{time lag (units of time).}$$

$$\frac{3}{P(i\omega)} \quad = \quad \text{complex gain} \quad = \frac{3}{|P(i\omega)|}e^{-i\phi} = ge^{-i\phi}$$

## 4.1 Gain and phase lag determine the output

Key: For sinusoidal input, the gain and phase lag determine the output.

Input $= B\cos(\omega t)$

Output $= g(\omega)B\cos(\omega t - \phi(w)) = gB\cos(\omega t - \phi)$

Together gain and phase lag is called the frequency response of the system.

See mathlet: `https://web.mit.edu/jorloff/www/OCW-ES1803/gainPhase.html`