Solutions Day 2, T 2/6/2024

Topic 1: Introduction to differential equations (day 2)

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Problem 1. *Solve* $\dfrac{dy}{dt} = ay$*, a a constant.*

Solution: Separate variables: $\dfrac{dy}{y} = a\,dt$.

Integrate both sides: $\ln|y| = at + c$.

Exponentiate: $|y| = e^c e^{at}$.

If $y < 0$, then $y = -e^c e^{at}$.
If $y > 0$, then $y = e^c e^{at}$.
If $y = 0$, this is the lost solution.

More simply, $y(t) = ce^{at}$, where c is an arbitrary constants.

Problem 2. *Check that* $y(t) = c_1\cos(2t) + c_2\sin(2t)$ *(c_1, c_2 constants) solves* $y'' + 4y = 0$.

What physical system does this model? (There are many possible answers.)

Solution: We plug $y(t) = c_1\cos(2t) + c_2\sin(2t)$ into the DE $y'' + 4y = 0$ and verify the equation is true. Going slowly

$$\begin{aligned} y &= c_1\cos(2t) + c_2\sin(2t) \\ y' &= -2c_1\sin(2t) + 2c_2\cos(2t) \\ y'' &= -4c_1\cos(2t) + -4c_2\sin(2t) \\ \text{So, } 4y &= 4c_1\cos(2t) + 4c_2\sin(2t) \end{aligned}$$

Adding the last two equations, we see $y'' + 4y = 0$. ■

The DE models a mass on a spring (Hooke's Law: $my'' = -ky$), i.e., simple harmonic motion.

Possible units are: y in meters, t in seconds, mass = 1 kg, $k = 4\,\text{kg/sec}^2$.

Problem 3. *Interpret Newton's law of cooling in words:* $T' = -k(T - E)$.

Solution: The DE says that the rate the temperature changes is proportional to the difference between the temperature of the body and that of its environment.

- k is a positive constant.
- The minus sign says that a body hotter than E cools and one colder than E warms.
- The greater the difference between the body and the environment, the faster the rate of change.

Problem 4. *Solve* $\dfrac{dy}{dt} = y^2$ *with initial value* $y(0) = 1$.

Graph the solution.

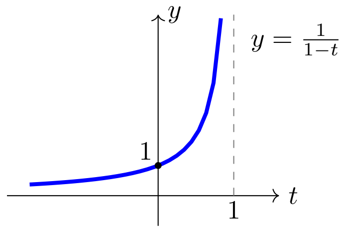
Solution: This is separable, $\dfrac{dy}{y^2} = dt$.

Integrating: $-\dfrac{1}{y} = t + c \quad \Rightarrow y = -\dfrac{1}{t+c}$.

The initial condition allows us to determine the value of c.

$$y(0) = 1 = -\frac{1}{c} \quad \Rightarrow c = -1.$$

So, $y(t) = -\dfrac{1}{t-1} = \dfrac{1}{1-t}$. This has a vertical asymptote at $t = 1$.

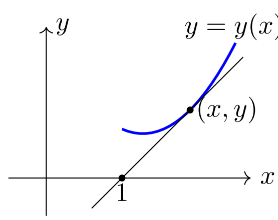


Note: We only consider the solution for $-\infty < t < 1$. As explained in the Topic 1 notes, we require solutions to be defined on a single interval, e.g., $(-\infty, 1)$. The function $y(t) = \dfrac{1}{1-t}$ for $1 < t < \infty$ is a solution to the DE, but it doesn't match the initial condition.

Because $y(t)$ goes to ∞ as t goes to 1, we say $y(t)$ blows up in a finite time.

Problem 5. *A curve $y = y(x)$ has the property that every tangent line goes through the point $(1,0)$. Find a DE for this curve. Solve the DE to find all curves with this property.*

Solution: The sketch shows a curve with the tangent line going through $(1,0)$.



Since $y = y(x)$, the slope of the tangent is $\dfrac{dy}{dx}$.

Since the tangent contains the points $(1,0)$ and (x,y), its slope is $\dfrac{y-0}{x-1}$.

Thus, $\boxed{\dfrac{dy}{dx} = \dfrac{y}{x-1}}$.

This is separable: $\dfrac{dy}{y} = \dfrac{dx}{x-1}$.

Integrate both sides: $\ln|y| = \ln|x-1| + c$.

The absolute values mean:
If $x > 1$, $y > 0$, then $y = \tilde{c}(x-1)$, with $\tilde{c} > 0$.
If $x > 1$, $y < 0$, then $y = \tilde{c}(x-1)$, with $\tilde{c} < 0$.
If $x < 1$, $y > 0$, then $y = \tilde{c}(x-1)$, with $\tilde{c} < 0$.
If $x < 1$, $y < 0$, then $y = \tilde{c}(x-1)$, with $\tilde{c} > 0$.

The lost solution is $y(x) \equiv 0$.

Putting this together, $y = \tilde{c}(x-1)$, where $\tilde{c}$ is an arbitrary constant.

That is, $y(x)$ is any line that passes through $(1,0)$.

Problem 6. *Suppose Oryx (African antelope) have a natural growth rate of $k = 0.02$/year (made up number). Suppose they are "harvested" at a rate of $h = 1000$ oryx/year.*

Model the population $x(t)$ by finding a DE from first principles using Δx and Δt.

How does your model change if $h = 10000\,\sin(2\pi t)$?

What is happening if $h < 0$?

Solution: Over a very small time interval $[t, t+\Delta t]$, the population changes:

Change due to natural growth (call it Δx_N): $\Delta x_N \approx kx\,\Delta t$.

Change due to harvesting (call it Δx_h): $\Delta x_h \approx -h\,\Delta t$.

Note: Δx_N is approximate because x is actually changing over the interval. If Δt is small, the error will be of the order $(\Delta t)^2$, so we can ignore it as $\Delta \to 0$.

Combining the changes, $\Delta x = \Delta x_N + \Delta x_h \approx kx\Delta t - h\Delta t$. Thus, $\dfrac{\Delta x}{\Delta t} \approx kx - h$. In the limit as $\Delta t \to 0$, we get

$$\frac{dx}{dt} = kx - h = 0.02x - 1000.$$

If h is a function of t, the derivation is unchanged. We get

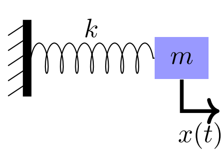
$$\frac{dx}{dt} = kx - h(t) = 0.02x - 1000\sin(2\pi t).$$

When h is negative, oryx are being added to the population. Negative harvesting is called stocking.

Problem 7. *Interpret Hooke's law, $m\dfrac{d^2x}{dt^2} = -kx$ in words.*

What is the dimension of k?

Solution: The equation says that, when displaced from equilibrium, the spring exerts a restoring force, i.e., the force is always directed towards the equibrium position. The magnitude of the force is proportional to displacement.



k has dimension mass/time^2, e.g., kg/sec^2.