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Topic 14: Row reduction (day 1)

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1 Agenda

- Row reduction
  - Echelon form
  - Reduced row echelon form (RREF)
  - Pivots
- Pivot and free variables
- Tomorrow
  - Column and null spaces – computation and meaning
  - Vocabulary: span, independence, basis, rank, dimension
  - Connection between A and RREF(A)
  - $A\mathbf{x} = \mathbf{b}$: solution = particular + homogeneous
  - View: matrix multiplication as a linear transformation

2 Row reduction and elimination

Example 1. Solve $\begin{array}{lll} 6x & +5y & =4 \\ x & +2y & =3 \end{array}$ by elimination.

Solution: Here are our steps. Each step leaves the solutions unchanged.

Swap equations	$\begin{cases} x + 2y = 3 \\ 6x + 5y = 4 \end{cases}$	
Subtract 6×Equation 1 from Equation 2	$\begin{cases} x + 2y = 3 \\ \quad - 7y = -14 \end{cases}$	Equation 1 is unchanged
Multiply Equation 2 by -1/7	$\begin{cases} x + 2y = 3 \\ \quad\quad y = 2 \end{cases}$	Equation 1 is unchanged
Subtract 2×Equation 2 from Equation 1	$\begin{cases} x \quad\quad = -1 \\ \quad\quad y = 2 \end{cases}$	Equation 2 is unchanged

Our solution is $\boxed{x = -1,\ y = 2}$.

Row reduction: Now, write the previous example in matrix form and solve using row reduction. Note that this is essentially identical to elimination.

Solution: In matrix form: $\begin{array}{rcrcr} 6x & + & 5y & = & 4 \\ x & + & 2y & = & 3 \end{array} \longrightarrow \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

Augmented matrix: $\left[\begin{array}{cc|c} 6 & 5 & 4 \\ 1 & 2 & 3 \end{array}\right]$.

Row reduction:

Swap rows	$\left[\begin{array}{cc\|c} 1 & 2 & 3 \\ 6 & 5 & 4 \end{array}\right]$	
Subtract 6·Row_1 from Row_2	$\left[\begin{array}{cc\|c} 1 & 2 & 3 \\ 0 & -7 & -14 \end{array}\right]$	Row_1 is unchanged
Scale Row_2 by $-1/7$	$\left[\begin{array}{cc\|c} 1 & 2 & 3 \\ 0 & 1 & 2 \end{array}\right]$	Row_1 is unchanged
Subtrace 2·Row_2 from Row_1	$\left[\begin{array}{cc\|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$	Row_2 is unchanged
Translate back to equations	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$	$x = -1$, $y = 2$

2.1 Elementary row operations

- Swap rows
- Add a multiple of one row to another
- Scale a row by a non-zero number

3 Row reduced echelon form (RREF)

Example 2. Let $A = \begin{bmatrix} \textcircled{2} & -1 & 2 & 2 \\ 4 & 1 & 16 & 2 \\ 4 & 1 & 16 & 1 \end{bmatrix}$

Use row reduction to put A into echelon form and then RREF.

Solution: We reduce moving down and right.

The 2 in the 1,1 entry is non-zero. It is our first pivot. We use it to eliminate the entries below.

Subtract 2·Row_1 from Row_2
Subtract 2·Row_1 from Row_3
$$\begin{bmatrix} \textcircled{2} & -1 & 2 & 2 \\ 0 & \textcircled{3} & 12 & -2 \\ 0 & 3 & 12 & -3 \end{bmatrix}$$

Next pivot = 3.

Subtract Row_2 from Row_3
$$\begin{bmatrix} \textcircled{2} & -1 & 2 & 2 \\ 0 & \textcircled{3} & 12 & -2 \\ 0 & 0 & 0 & \textcircled{-1} \end{bmatrix}$$
This is in echelon form.

No pivot in Column 3. Next pivot is -1.

This is in echelon form: all entries to the left and below the pivots are 0.

We continue the row reduction to row reduced echelon form (RREF). For this, we want all the pivots to be 1 and columns with pivots should be all zeros (except for the pivot).

We start at the bottom right and work our way up and to the left.

Make the last pivot 1:	Scale Row_3 by -1	$\begin{bmatrix} 2 & -1 & 2 & 2 \\ 0 & 3 & 12 & -2 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$
Eliminate above the pivot:	Add $2{\cdot}\text{Row}_3$ to Row_2 Subtract $2{\cdot}\text{Row}_3$ from Row_2	$\begin{bmatrix} 2 & -1 & 2 & 0 \\ 0 & 3 & 12 & 0 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$
Make the next pivot 1:	Scale Row_2 by 1/3	$\begin{bmatrix} 2 & -1 & 2 & 0 \\ 0 & \textcircled{1} & 4 & 0 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$
Eliminate above the pivot:	Add Row_2 to Row_1	$\begin{bmatrix} 2 & 0 & 6 & 0 \\ 0 & \textcircled{1} & 4 & 0 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$
Make the first pivot 1:	Scale Row_1 by 1/2	$\begin{bmatrix} \textcircled{1} & 0 & 3 & 0 \\ 0 & \textcircled{1} & 4 & 0 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix} = R$

$$R = \begin{bmatrix} \textcircled{1} & 0 & 3 & 0 \\ 0 & \textcircled{1} & 4 & 0 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix} \text{ is in RREF.}$$

- Pivots = 1
- Columns with pivots are all zeros except for the pivot
- To the left of a pivot the row is all zeros
- Every row has a pivot or else is all zeros
- Every all zero row is at the bottom

3.1 Pivot and free columns, pivot and free variables

For R in RREF, the pivot columns are the ones with pivots, the others are called free columns. In the example above we have

$$A = \begin{bmatrix} 2 & -1 & 2 & 2 \\ 4 & 1 & 16 & 2 \\ 4 & 1 & 16 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} \textcircled{1} & 0 & 3 & 0 \\ 0 & \textcircled{1} & 4 & 0 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$$

Pivot columns   Free column   Pivot columns   Free column

The pivot and free columns of A are, by definition, the same as those of R.

Rank of A = number of pivots = 3.

Pivot and free variables:

Writing matrix multiplication as a linear combination of the columns, we have.

Pivot variable Free variable Pivot variable

$$\begin{bmatrix} 2 & -1 & 2 & 2 \\ 4 & 1 & 16 & 2 \\ 4 & 1 & 16 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 16 \\ 16 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Free column

Pivot column Pivot column

Pivot variables go with pivot columns.

Free variables go with free columns.

3.2 Relationship between free and pivot columns

In R, it's easy to see that $\text{Col}_3 = 3\text{Col}_1 + 4\text{Col}_2$.

It's also easy to check that the columns of A have the same relationship. That is, row reduction does not change the relationship between the columns.

Conclusion: For both A and R, the free columns are (the same) linear combinations of the pivot columns.