Solutions Day 30, M 3/18/2024

Topic 14: Row reduction (day 1)

Jeremy Orloff

Problem 1. *Let* $A = \begin{bmatrix} 1 & 2 & 2 & 11 \\ 2 & 4 & 1 & 10 \\ 3 & 6 & 0 & 9 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) *Use row reduction to show that $R = RREF(A)$.*

Solution: Let R_1 be Row 1, etc.

$$A = \begin{bmatrix} \textcircled{1} & 2 & 2 & 11 \\ 2 & 4 & 1 & 10 \\ 3 & 6 & 0 & 9 \end{bmatrix} \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} \textcircled{1} & 2 & 2 & 11 \\ 0 & 0 & \textcircled{-3} & -12 \\ 0 & 0 & -6 & -24 \end{bmatrix} \xrightarrow{R_3 = -\frac{1}{3} \cdot R_3} \begin{bmatrix} \textcircled{1} & 2 & 2 & 11 \\ 0 & 0 & \textcircled{1} & 4 \\ 0 & 0 & -6 & -24 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 + 6R_2} \begin{bmatrix} \textcircled{1} & 2 & 2 & 11 \\ 0 & 0 & \textcircled{1} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 - 2R_2} \begin{bmatrix} \textcircled{1} & 2 & 0 & 3 \\ 0 & 0 & \textcircled{1} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R \quad \blacksquare$$

(b) *Identify the free and pivot variables of R.*

Solution: Let C_1 be Column 1, etc.

Pivot columns: C_1, C_3. Free columns: C_2, C_4.

(c) *What is rank(A)?*

Solution: Rank = number of pivots = 2.

(d) *Give the relations between the free and pivot columns of R. (That is, write each free column as a linear combination of the pivot columns.)*

Solution: It's easy to see in R that $C_2 = 2C_1$, $C_4 = 3C_1 + 4C_3$.

(e) *Verify that the columns of A have the same relations as in R.*

Solution: Check $C_2 = 2C_1$: $2C_1 = 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = C_2 \quad \blacksquare$.

Check $C_4 = 3C_1 + 4C_3$: $3C_1 + 4C_3 = 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \\ 9 \end{bmatrix} = C_4 \quad \blacksquare$.

(f) *Use the relations between the columns of R to find a vector $\mathbf{v}$ such the $R\mathbf{v} = \mathbf{0}$.*

Solution: $C_2 = 2C_1 \quad \Rightarrow -2C_1 + C_2 = \mathbf{0}$. Viewing matrix multiplication as a linear combination of the columns, we get $R \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So, $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution to $R\mathbf{v} = \mathbf{0}$.

Likewise, $C_4 = 3C_1 + 4C_3 \quad \Rightarrow -3C_1 - 4C_3 + C_4 = \mathbf{0}$. So, $\mathbf{v} = \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$ solves $R\mathbf{v} = \mathbf{0}$.

(g) *Verify that $A\mathbf{v} = \mathbf{0}$. ($\mathbf{v}$ your answer to the previous part)*

Solution: Since A has the same relationships between its columns as R, the same logic as in Part (f), will produce the same vectors. That is,

$$A \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = -2C_1 + C_2 = \mathbf{0}, \qquad A \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} = -3C_1 - 4C_3 + C_4 = \mathbf{0}.$$

(h) *Find a solution to $A\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ by setting the free variabes to 0 and solving the resulting 3×2 system using row reduction on the augmented matrix.*

Solution: Setting the free variables to 0 means letting $x_2 = 0$, $x_4 = 0$. So we have to solve

$$A \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \qquad \Rightarrow x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}.$$

In matrix form this is $\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$. So the augmented matrix is $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 3 & 0 & 6 \end{array}\right]$.

The row reduction steps are

$$\xrightarrow{\substack{R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1}} \left[\begin{array}{cc|c} \textcircled{1} & 2 & 0 \\ 0 & \textcircled{-3} & 3 \\ 0 & -6 & 6 \end{array}\right] \xrightarrow{R_2 = -\frac{1}{3}R_2} \left[\begin{array}{cc|c} \textcircled{1} & 2 & 0 \\ 0 & \textcircled{1} & -1 \\ 0 & -6 & 6 \end{array}\right] \xrightarrow{R_3 = R_3 + 6R_2} \left[\begin{array}{cc|c} \textcircled{1} & 2 & 0 \\ 0 & \textcircled{1} & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{cc|c} \textcircled{1} & 0 & 2 \\ 0 & \textcircled{1} & -1 \\ 0 & 0 & 0 \end{array}\right]$$

Back in matrix form: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \Rightarrow x_1 = 2,\ x_3 = -1.$

Thus we have a particular solution $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$.