Day 4, R 2/8/2024

Topic 2: Linear Systems
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1 Agenda

- For tomorrow, read Topic 3 notes
- First-order linear DEs
- Variation of parameters formula
- Superposition principle (also called linearity)

2 First-order linear DEs

General form: $A(t)y' + B(t)y = C(t)$

Standard form: $y' + p(t)y = q(t)$

- Right now we identify linear equations by their form.
  Later, we will use their properties.
- Sometimes call $q(t)$ "input" to the DE
- Always work with standard form

$$\begin{array}{lll} \text{(I)} & y' + p(t)y = q(t) & \text{(inhomogeneous DE)} \\ \text{(H)} & y' + p(t)y = 0 & \text{(homogeneous DE)} \end{array}$$

Do problem 1

2.1 Dot notation

$\dfrac{dy}{dt} = y' = \dot{y}$ all mean the same thing.
The dot is reserved for derivatives with respect to time.

3 Variation of parameters formula

Solution to (H): $y_h(t) = e^{-\int p(t)\,dt}$ (separable DE)

Solution to (I): $y(t) = y_h(t)\left[\int \dfrac{q(t)}{y_h(t)}\,dt + C\right] = y_h(t)\int \dfrac{q(t)}{y_h(t)}\,dt + Cy_h(t).$

Proof: In Topic 2 notes – very pretty.

Examples in problems.

4 Superposition principle (= linearity)

4.1 Linear combinations (also called superposition)

If f_1, f_2 are functions and c_1, c_2 are constants then

$$f(t) = c_1 f_1(t) + c_2 f_2(t)$$

is called a linear combination of f_1 and f_2.

Superposition principle

$$\left.\begin{array}{llll} \text{If} & y_1 \text{ solves} & y' + p(t)y & = q_1(t) \\ \text{and} & y_2 \text{ solves} & y' + p(t)y & = q_2(t) \end{array}\right\} \text{ same } p(t)\text{, different } q(t)$$

then, for any constants c_1, c_2

$$y = c_1 y_1 + c_2 y_2 \quad \text{solves} \quad y' + p(t)y = c_1 q_1 + c_2 q_2$$

In words, a linear combination of the inputs produces a linear combination of the solutions.

- The hypothesis y_1 solves $y' + p(t)y = q_1(t)$ can also be expressed as $y_1' + p(t)y_1 = q_1(t)$. (Likewise for y_2.)
- Superposition/linearity is ubiquitous and super-important throughout math, science and engineering.
- It is **always** easy to check if linearity holds. (You just have to think to ask.)

See extended Examples 2.6, 2.7 in the Topic 2 notes.

4.2 Other familiar places where linearity holds

Integrals: $\displaystyle\int c_1 f_1(x) + c_2 f_2(x)\,dx = c_1 \int f_1(x)\,dx + c_2 \int f_2(x)\,dx.$

Multiplication: $a(c_1 x_1 + c_2 x_2) = c_1 a x_1 + c_2 a x_2.$

Matrix multiplication: $M(c_1 \mathbf{v_1} + c_2 \mathbf{v_2}) = c_1 M \mathbf{v_1} + c_2 M \mathbf{v_2}.$

4.3 Examples: incorrectly assuming linearity

$$\sqrt{a^2 + b^2} \neq a + b$$
$$\frac{1}{a+b} \neq \frac{1}{a} + \frac{1}{b}$$

(Assuming these are equalities is incorrectly assuming linearity.)