Solutions Day 44, F 4/12/2024

Topic 21: Fourier series (day 2)

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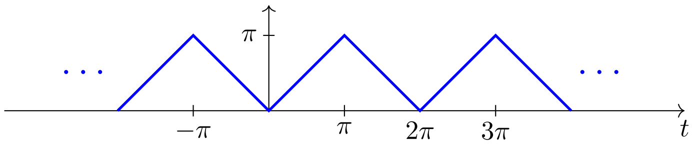
Note: There is a useful integral table on the last page.

Problem 1. $\text{tri}(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}$.



Graph of $\text{tri}(t)$

(a) *Convince yourself that* $\text{tri}'(t) = \text{sq}(t)$.

Solution: $\text{tri}(t)$ alternates between slopes of 1 and -1.

Looking at the graph over one period, $\text{tri}'(t) = \begin{cases} -1 & \text{for } -\pi < t < 0 \\ 1 & \text{for } 0 < t < \pi \end{cases}$.

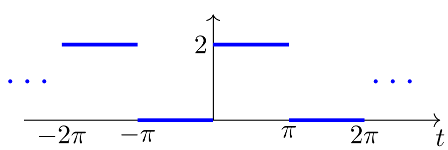
This last expression is the same as $\text{sq}(t)$ over the one period: $-\pi < t < \pi$.

(b) *Check that the derivative of the Fourier series for* $\text{tri}(t)$ *equals the Fourier series of* $\text{sq}(t)$.

Solution: $\text{tri}(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}$. So,

$$\text{tri}'(t) = -\frac{4}{\pi} \sum_{n \text{ odd}} \frac{-n\sin(nt)}{n^2} = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n} = \text{sq}(t) \qquad \blacksquare$$

Problem 2. *Let* $f(t) = 1 + \text{sq}(t)$. *Find the Fourier series for* $f(t)$.



Graph of $f(t)$

Solution: $\text{sq}(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}$.

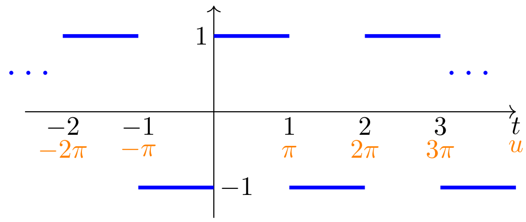
So, $f(t) = 1 + \text{sq}(t) = 1 + \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}$.

This is a Fourier series, so it must be the Fourier series for $f(t)$.

Problem 3. *Let* $g(t)$ *have period 2 and* $g(t) = \begin{cases} -1 & \text{for } -1 < t < 0 \\ 1 & \text{for } 0 < t < 1. \end{cases}$

Find the Fourier series for $g(t)$.

Solution: The graph of $g(t)$ is



Graph of $g(t)$

First we show that $g(t) = \text{sq}(\pi t)$. You can see this by letting $u = \pi t$ and adding the u scale below the t scale. Now it's clear that $g(t) = \text{sq}(u) = \text{sq}(\pi t)$.

Therefore, $g(t) = \dfrac{4}{\pi} \displaystyle\sum_{n \text{ odd}} \frac{\sin(n\pi t)}{n}$.

As a check, note that each of the terms $\sin(n\pi t)$ has period 2. This is the same as the period of $g(t)$.

Integrals *(for n a positive integer)*

1. $\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$

$1'.$ $\int_0^{\pi} t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n}.$

2. $\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$

$2'.$ $\int_0^{\pi} t\cos(nt)\,dt = \begin{cases} \frac{-2}{n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \neq 0 \text{ even} \end{cases}$

3. $\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$

$3'.$ $\int_0^{\pi} t^2\sin(nt)\,dt = \begin{cases} \frac{\pi^2}{n} - \frac{4}{n^3} & \text{for } n \text{ odd} \\ \frac{-\pi^2}{n} & \text{for } n \neq 0 \text{ even} \end{cases}$

4. $\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$

$4'.$ $\int_0^{\pi} t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2}$

If $a \neq b$

5. $\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

6. $\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

7. $\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$

8. $\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$

9. $\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$

10. $\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$