Day 54, M 4/29/2024

Topic 27: Linear phase portraits (day 1)
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1 Agenda

- Qualitative view of linear systems $\mathbf{x}' = A\mathbf{x}$
- Phase plane (xy-plane) and phase portraits
- Summary of types of critical points at 0
  – Main cases
  – Edge cases
  – Applet: https://mathlets.org/mathlets/linear-phase-portraits-matrix-entry/
- Drawing phase portraits

2 Qualitative view of linear systems

Example 1. Say $\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$. Suppose we find the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-7t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then

- $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a trivial solution (i.e., $c_1 = 0$, $c_2 = 0$).
  – Equilibrium (constant in time).
  – Critical point: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
- All solutions go asymptotically to 0 (negative exponents).
  – All solutions go asymptotically to the equilibrium.
  – In this case, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is called a dynamically stable equilibium or dynamically stable critical point.
- The system is qualitatively the same as one with the solution

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Key: Quality is determined by the eigenvalues.

3 Phase portraits

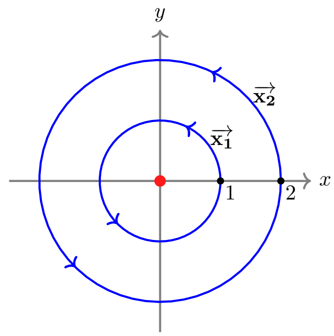
Phase plane $= xy$-plane.

A trajectory = graph of a solution $(x(t), y(t))$ as a parametrized curve in the plane.

Example 2. Suppose $\mathbf{x_1}(t) = \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$, $\mathbf{x_2}(t) = 2\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$, $\mathbf{x_3}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Sketch the 3 trajectories.

Solution:



Trajectories of $\mathbf{x_1}(t)$, $\mathbf{x_2}(t)$, $\mathbf{x_3}(t)$

4 Main types of critical points by eigenvalues

The name describes the equilibrium at (0,0).

$\lambda_1 \neq \lambda_2$,	both real, positive	$(0,0)$ = nodal source	(dynamically unstable equilibrium)
$\lambda_1 \neq \lambda_2$,	both real, negative	$(0,0)$ = nodal sink	(dynamically stable equilibrium)
$\lambda_1 < 0$, $\lambda_2 > 0$,	both real	$(0,0)$ = saddle	(dynamically unstable equilibrium)
$\lambda = \alpha + i\beta$,	$\alpha > 0$, $\beta \neq 0$	$(0,0)$ = spiral source	(dynamically unstable equilibrium)
$\lambda = \alpha + i\beta$,	$\alpha < 0$, $\beta \neq 0$	$(0,0)$ = spiral sink	(dynamically stable equilibrium)

Show applet: `https://mathlets.org/mathlets/linear-phase-portraits-matrix-entry/`

5 Drawing phase portraits

Phase portrait: Draw enough trajectories to get the idea. Always include the equilibium solution.

Example 3. Suppose the solution to $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Sketch a phase portrait.

Solution: From the table above, λ are positive and unequal $\longrightarrow$ nodal source.

The final sketch is below. Here are the steps.

Step 1: Sketch the equilibrium solution: $\mathbf{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ = single point.

Step 2: Sketch the modes:

Modal solutions: $\mathbf{x_1}(t) = c_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{x_2}(t) = c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Mode $e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$: trajectory = positive x-axis.

Mode $-e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$: trajectory = negative x-axis.

Likewise, the trajectories of $\mathbf{x}(t) = e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{x}(t) = -e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the positive and negative y-axis.

Summary: modes give straight line trajectories.

Step 3: Sketch some "mixed modal" solutions, e.g., $\mathbf{x}(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

<u>Asymptotics as $t \to \infty$</u>: As $t \to \infty$, $\mathbf{x}(t)$ goes to infinity.

$$\mathbf{x}'(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{2t} \left(e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

So $\mathbf{x}'(t)$ is parallel to $e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

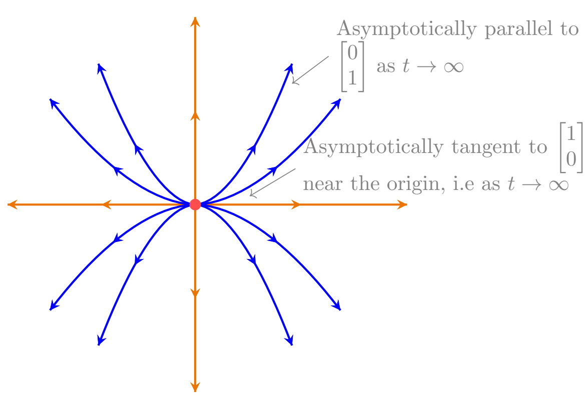
So it is asymptotically parallel to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, i.e., asymptotically parallel to the mode with the bigger λ.

<u>Asymptotics as $t \to -\infty$</u>: As $t \to -\infty$, $\mathbf{x}(t)$ goes to zero.

$$\mathbf{x}'(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{t} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

So $\mathbf{x}'(t)$ is parallel to $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

So it is asymptotically parallel to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, i.e., asymptotically parallel to the mode with the smaller λ.

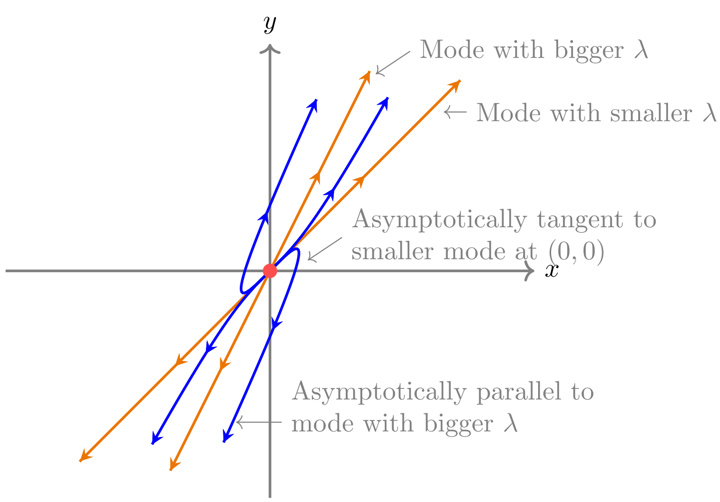


Nodal source at $(0, 0)$ – eigenvalues positive, different.
All trajectories "flow out" from the origin.

Key points

- Trajectories don't cross
- They fill up the plane
- Different solutions with the same trajectory have different initial values, e.g., $\mathbf{x_1}(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{x_2}(t) = 2e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ have the same trajectory, but $\mathbf{x_1}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{x_2}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ are different initial values.
- For nodal sources:
  - As $t \to \infty$, $\mathbf{x}(t) \to \infty$ and trajectories become parallel to the mode with the bigger λ.
  - As $t \to -\infty$, $\mathbf{x}(t) \to \mathbf{0}$ and trajectories become tangent to the mode with the smaller λ.
  - Positive, different eigenvalues give the same qualitative picture, i.e., a nodal source.

Here is a phase portrait for a system with solution $\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Qualitatively, it has the same phase portrait as our previous example.



Nodal source at $(0, 0)$