Day 6, M 2/12/2024

Topic 4: Complex numbers (day 1 of 2)
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1 Agenda

- Complex numers: terminology, arithmetic, Euler's formula

2 Terminology

Define i by $i^2 = -1$. (imaginary i).

Complex numbers are of the form $z = x + iy$, where x and y are real numbers.

C = set of all complex numbers.

If $z = x + iy$:

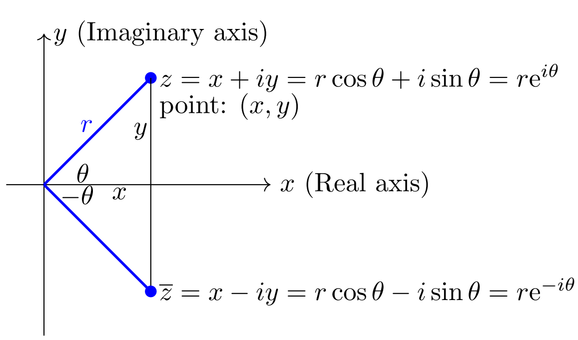
$\text{Re}(z) = x =$	'real part of z'
$\text{Im}(z) = y =$	'imaginary part of z' (No i in $\text{Im}(z)$.)
$\overline{z} = x - iy =$	'complex conjugate of z = 'z bar'
$\lvert z\rvert = \sqrt{x^2 + y^2}\rvert =$	magnitude of z (also: modulus, norm absolute value) Note: no i on the y.

Example 1. $\text{Re}(2+3i) = 2$, $\text{Im}(2+3i) = 3$, $|2+3i| = \sqrt{13}$, $\overline{2+3i} = 2 - 3i$.

3 Arithmetic

Uses $i^2 = -1$. Practice with problems.

4 Complex plane



r, θ = usual polar coordinates: $r = |z|$, $\theta = \text{Arg}(z)$.

5 Euler's formula

Euler's formula for complex exponentials is

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{*** Important ***}$$

In the figure above we have

$$z = \underbrace{x + iy}_{\text{rectangular form}} = r\cos\theta + ir\sin\theta = \underbrace{re^{i\theta}}_{\text{polar form}}.$$

The second equation is simple trigonometry. The third is Euler's formula.

For $z = re^{i\theta}$: magnitude $|z| = r$, argument $= \text{Arg}(z) = \theta$.

5.1 Verify e^{it} behaves like an exponential

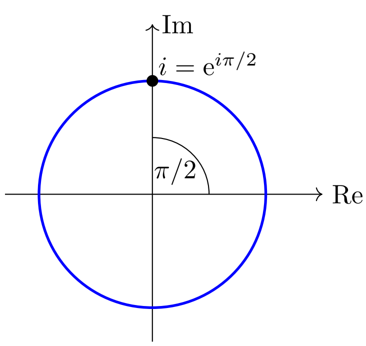
1. $e^{i0} = \cos(0) + i\sin(0) = 1$ ✓
2. $\frac{d}{dt}e^{it} = \frac{d}{dt}(\cos t + i\sin t) = -\sin t + i\cos t = i(\cos t + i\sin t) = ie^{it}$ ✓
3. $e^{i\alpha + i\beta} = e^{i\alpha}e^{i\beta}$ (Trig identities. See Topic 4 notes) ✓
4. Taylor series ✓

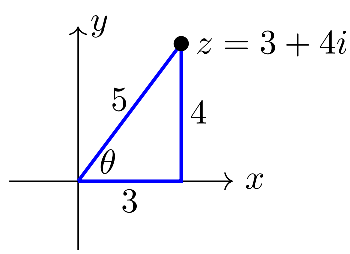
Key facts: $|e^{i\theta}| = 1$, $e^{i2\pi} = 1$, $e^{i\pi/2} = i$.

Argument: Can always add multiples of 2π to θ.

Example 2. $|z|$ and $\text{Arg}(z)$:

$z = 2$	$\lvert z\rvert = 2$	$\text{Arg}(z) = 0, 2\pi, 4\pi, \ldots, 2n\pi$, n any integer
$z = i$	$\lvert z\rvert = 1$	$\text{Arg}(z) = \frac{\pi}{2} + 2n\pi$ (See figure below)
$z = 3 + 4i$	$\lvert z\rvert = \sqrt{3^2 + 4^2} = 5$	$\text{Arg}(z) = \tan^{-1}(4/3)$ in Q1. (See figure below.)





Example 3. Complex replacement

Compute $I = \int e^{2x}\cos(3x)\,dx$.

– In 1803 we don't really care about this integral.

– We do really care about the following technique used to compute it. (We'll use the same technique to solve important DEs in a few days.)

Solution: Short form of the solution. Reasons below and in Topic 4 notes.

Replace $\cos(3x)$ by e^{3ix}. Note, $\cos(3x) = \text{Re}\left(e^{3ix}\right)$.

So, $I = \int e^{2x}\cos(3x)\,dx$ becomes

$$I_c = \int e^{2x}e^{3ix}\,dx = \int e^{(2+3i)x}\,dx, \qquad I = \text{Re}(I_c).$$

Integrating: $I_c = \dfrac{e^{(2+3i)x}}{2+3i}$.

We need the real part of I_c. In 1803 we almost always use the polar form.

$$|2+3i| = \sqrt{13}, \quad \boxed{\phi = \text{Arg}(2+3i) = \tan^{-1}(3/2) \text{ in Q1.}}$$

Thus, $2+3i = \sqrt{13}\,e^{i\phi}$. So, $I_c = \dfrac{e^{2x}e^{3ix}}{\sqrt{13}\,e^{i\phi}} = \dfrac{e^{2x}}{\sqrt{13}}\,e^{i(3x-\phi)} = \dfrac{e^{2x}}{\sqrt{13}}\left(\cos(3x-\phi)+i\sin(3x-\phi)\right)$.

So, $\boxed{I = \text{Re}(I_c) = \dfrac{e^{2x}}{\sqrt{13}}\cos(3x-\phi)}$.

Justification of complex replacement. The trick comes by cleverly adding a new integral to I as follows. Let $J = \int e^{2x}\sin(3x)\,dx$. Then we let

$$I_c = I + iJ = \int e^{2x}(\cos(3x) + i\sin(3x))\,dx = \int e^{2x}e^{3ix}\,dx.$$

Clearly, $\text{Re}(I_c) = I$ as claimed above.

6 Next time

Fundamental Theorem of Algebra: A polynomial of degree n has **exactly** n complex roots.

Finding roots of a polynomial: Uses Euler's formula.