Day 7, T 2/13/2024

Topic 4: Complex numbers (day 2 of 2)

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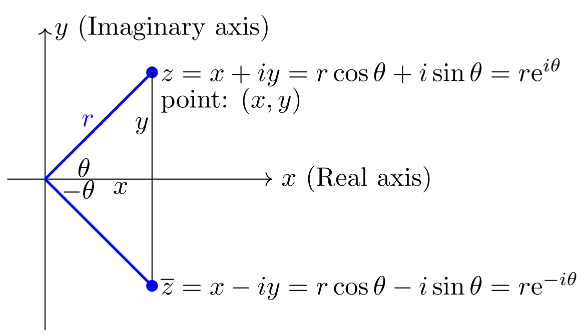
1 Agenda

- For next time read Topic 5 notes
- Finish plan from yesterday
- n^{th} roots
- Problems from yesterday

2 Review

- $z = x + iy = re^{i\theta}$
- $e^{i\theta} = \cos\theta + i\sin\theta$
- $|z| = \sqrt{x^2 + y^2} = r, \qquad \text{Arg}(z) = \theta + 2\pi n.$



Complex relacement: Key: $\text{Re}(e^{iat}) = \cos(at)$, $\text{Im}(e^{iat}) = \sin(at)$.

3 Fundamental theorem of algebra

A polynomial of degree n has **exactly** n complex roots.

Example 1. $P(x) = (x-1)(x-4)^2(x-6)^3(x^2+1)$ has roots

$$1,\ 4,\ 4,\ 6,\ 6,\ 6,\ i,\ -i.$$

We say 6 is a root of multiplicity 3.

3.1 n^{th} roots

Example 2. Find all the cube roots of $12i$.

Solution: This asks us to find values z such that $z^3 = 12i$. That is, find the roots of $z^3 - 12i = 0$.

By the fundamental theorem there are exactly 3 roots to find. We work in polar form:

$$|12i| = 12, \qquad \underbrace{\text{Arg}(12i) = \frac{\pi}{2} + 2n\pi}_{\text{It's important to include all values of } \theta}$$

So, $12i = 12e^{i(\pi/2+2n\pi)}$. This implies,

$$(12i)^{1/3} = 12^{1/3}e^{i(\pi/6+2n\pi/3)}$$

That is,

$$(12i)^{1/3} = \underbrace{12^{1/3}e^{i\pi/6}}_{n=0}, \quad \underbrace{12^{1/3}e^{i5\pi/6}}_{n=1}, \quad \underbrace{12^{1/3}e^{i9\pi/6}}_{n=2}, \quad \underbrace{12^{1/3}e^{i13\pi/6}}_{n=3}, \quad \ldots$$

Since $\dfrac{13\pi}{6} = \dfrac{\pi}{6} + 2\pi$, the $n = 0$ and $n = 3$ values are the same. Likewise, for $n = 1$, $n = 4$, etc. So we have 3 distinct roots:

$$\begin{aligned}
12^{1/3}e^{i\pi/6} &= 12^{1/3}\left(\frac{\sqrt{13}}{2} + \frac{1}{2}\,i\right) \\
12^{1/3}e^{i5\pi/6} &= 12^{1/3}\left(\frac{-\sqrt{13}}{2} + \frac{1}{2}\,i\right) \\
12^{1/3}e^{i9\pi/6} &= 12^{1/3}\,i
\end{aligned}$$

(We computed the complex exponentials by thinking about 30-60-90 triangles.)