18.03: Existence and Uniqueness Theorem

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1 Introduction

The existence and uniqueness theorem for differential equations is a key technical result. For example, when we solve an equation like $x'' + 8x' + 7x = 0$, we first find the modal solutions $x_1(t) = e^t$, $x_2(t) = e^{7t}$. Then we claim that the general solution is the set of all linear combinations, i.e., $x(t) = c_1e^t + c_2e^{7t}$. The algebra leading to this makes it clear that $x(t)$ is a solution, but it does not show that this is all the solutions. To show that we need the existence and uniqueness theorem.

The analysis needed in the proof of the theorem is beyond what we can do in ES.1803. But, the proof using Picard iteration is quite beautiful and we can give an outline which will give you a sense of how one goes about proving something like this.

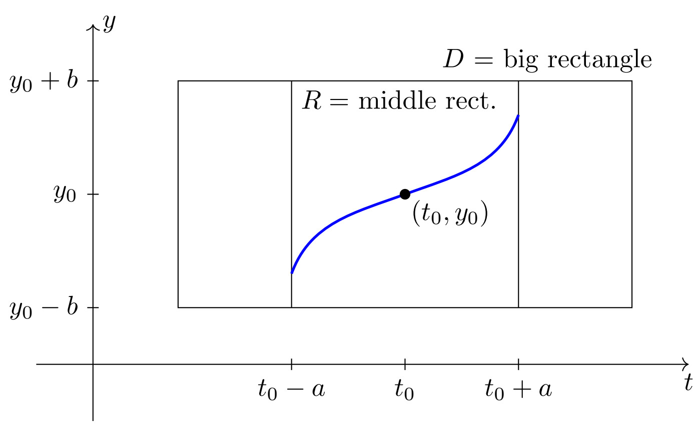
2 Statement of the theorem

Theorem (Existence and uniqueness)

Suppose $f(t, y)$ and $\dfrac{\partial f}{\partial y}(t, y)$ are continuous on the rectangle D as shown. Suppose also that (t_0, y_0) is in D and we have the IVP

$$\frac{dy}{dt} = f(t, y(t)), \quad y(t_0) = y_0$$

Then, we can choose a smaller rectangle R (as shown) so that the IVP has a unique solution, which is defined on $[t_0 - a, t_0 + a]$ and whose graph is entirely inside R.



3 Proof

The proof proceeds in a series of steps. Some of these steps are technical –I'll try to give a sense of why they are true. The key steps are the definition of the contraction map T (Step 3) and the use of T in Picard iteration (Step 8).

Step 1: Lipshcitz condition

Let $M = \max_D |f(t,y)|$ and $L = \max_D \left|\frac{\partial f}{\partial y}(t,y)\right|$.

If (t,y_1) and (t,y_2) are in D, then the mean value theorem implies $f(t,y_2) - f(t,y_1) = \frac{\partial f}{\partial y}(t,c)\,(y_2 - y_1)$ (for some c between y_1 and y_2). Thus,

$$|f(t,y_2) - f(t,y_1)| < L\,|y_2 - y_1| \quad \text{(Lipshcitz condition).}$$

Step 2: Choosing the rectangle R

Choose $a < \min(\frac{b}{M}, \frac{1}{2L})$. This defines the rectangle R (see above figure). We will use this in steps 3 and 5.

Step 3: The operator T

Let $\mathbf{Y}$ be the space of all functions $y(t)$ which are continuous on $[t_0 - a, t_0 + a]$ and whose graph is entirely inside R. For any $y \in \mathbf{Y}$ define

$$Ty = z(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

We note a number of easy facts about T.

(a) $Ty = z(t)$ is well defined on $[t_0 - a, t_0 + a]$. (Proof: $(s, y(s))$ is in R, so the integrand $f(s, y(s))$ is defined and continuous.)

(b) $z(t)$ is continuous. (Proof: trivial since both y and f are continuous.)

(c) The graph of $z(t)$ is entirely in R.

Proof: $|z(t) - y_0| = \left|\int_{t_0}^{t} f(s, y(s))\, ds\right| \le M\,|t - t_0| \le Ma < b$. (The last inequality follows from the choice of a in Step 2.)

(d) $z'(t) = f(t, y(t))$ (Proof: fundamental theorem of calculus).

Facts a-c show T maps the space $\mathbf{Y}$ into itself.

Definition: For the function $y \in \mathbf{Y}$, if $Ty = y$, then y is called a *fixed point* of T.

Claim: y is a solution to the IVP $\Leftrightarrow$ y is a fixed point of T.

Proof: Suppose y is a solution, i.e., $y(t_0) = y_0$ and $y'(t) = f(t, y(t))$. We have:

$$\begin{aligned} Ty &= y_0 + \int_{t_0}^{t} f(s, y(s))\, ds \\ &= y_0 + \int_{t_0}^{t} y'(s)\, ds = y_0 + y(s)\big|_{t_0}^{t} = y(t). \end{aligned}$$

So y is a fixed point of T.

Conversely, suppose y is a fixed point, then $y = Ty = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$.

This implies $y(t_0) = y_0$ and $y' = f(t, y(t))$, i.e., y satisfies the IVP. QED

The claim shows that proving existence and uniqueness is equivalent to proving that T has a unique fixed point. (This is proved in steps 8 and 9 below.)

Step 4: The metric on Y

For y_1 and y_2 in $\mathbf{Y}$ define

$$\delta(y_1, y_2) = \max_{[t_0-a,t_0+a]} |y_1(t) - y_2(t)|.$$

δ is called a metric on $\mathbf{Y}$. We have the following facts about δ.

(a) $\delta(y_1, y_2) = 0 \Leftrightarrow y_1 = y_2$ (Proof: trivial).

(b) δ satisfies the triangle inequality: $\delta(y_1, y_2) + \delta(y_2, y_3) \geq \delta(y_1, y_3)$ (Proof: not hard).

(c) δ tells how to measure 'closeness' between 'points' of $\mathbf{Y}$.

(d) (Technical statement) $\mathbf{Y}$ is a *complete* metric space. That is, all *Cauchy sequences* in $\mathbf{Y}$ converge to a function in $\mathbf{Y}$.

It is enough for us to know that this implies the following: If the sequence y_0, y_1, ... satisfies $\sum \delta(y_{n+1}, y_n) < \infty$ then the sequence converges, i.e., $\lim_{n\to\infty} y_n = y$ exists.

Completeness is not hard to show. It does require a careful '$\epsilon - \delta$' proof.

Step 5: Claim: $\delta(Ty_1, Ty_2) \leq \frac{1}{2}\delta(y_1, y_2)$.

Proof:

$$\begin{aligned}
|Ty_1(t) - Ty_2(t)| &= \left| \int_{t_0}^{t} f(s, y_1(s)) - f(s, y_2(s))\, ds \right| \\
&\leq \int_{t_0}^{t} |f(s, y_1(s)) - f(s, y_2(s))|\, ds \\
&\leq L \int_{t_0}^{t} |y_1(s) - y_2(s)|\, ds \qquad \text{(Lipschitz condition)} \\
&\leq L\delta(y_1, y_2) \int_{t_0}^{t} ds \qquad \text{(pull out } \max(y_1(s) - y_2(s))\text{)} \\
&= L\delta(y_1, y_2)(t - t_0) \\
&\leq \delta(y_1, y_2)\, L \cdot a \\
&< \frac{1}{2}\delta(y_1, y_2) \qquad \text{QED}
\end{aligned}$$

The last inequality uses the choice of a in Step 2.

Note: since T shrinks distances it is called a contraction mapping.

Step 6: Claim: T has at most one fixed point.

Proof: Suppose there were two different fixed points $y_1$and y_2. Then since $Ty_j = y_j$ we get $\delta(Ty_1, Ty_2) = \delta(y_1, y_2)$. But this contradicts Step 5, where we saw $\delta(Ty_1, Ty_2) \leq \frac{1}{2}\delta(y_1, y_2)$.

Step 7: If the sequence y_0, y_1, y_2, ... converges to y then Ty_0, Ty_1, Ty_2, ... converges to Ty.

Formally: T is a continuous map of $\mathbf{Y}$ to itself.

Step 8: Picard itereation

Start with $y_0(t) = y_0$. Let $y_1 = Ty_0$, $y_2 = Ty_1 = T^2 y_0$, ..., $y_{n+1} = Ty_n = T^n y_0$.

Claim: The sequence y_0, y_1, ... converges.

Proof: $\delta(y_2, y_1) = \delta(Ty_1, Ty_0) \le \frac{1}{2}\delta(y_1, y_0)$.

Likewise, $\delta(y_3, y_2) = \delta(Ty_2, Ty_1) \le \frac{1}{2}\delta(y_2, y_1) \le \frac{1}{4}\delta(y_1, y_0)$.

Generally, $\delta(y_{n+1}, y_n) \le \left(\frac{1}{2}\right)^n \delta(y_1, y_0)$. So, $\sum_{n=0}^{\infty} \delta(y_{n+1}, y_n) \le \delta(y_1, y_0) \sum_0^{\infty} \left(\frac{1}{2}\right)^n$.

Since this last sum converges, so the completeness of $\mathbf{Y}$ proves the claim.

Step 9: Take the sequence from Step 8 and let $y = \lim_{n\to\infty} y_n$.

Claim: y is a fixed point of T.

Proof: Since $y = \lim T^n y_0$, we have $Ty = \lim T^{n+1} y_0 = y$. QED

We have now proved the existence and uniqueness theorem. That is, we know that all solutions are fixed points of T and we have shown that T has a unique fixed poont.

4 Example of Picard iteration

Example: (Picard iteration) Consider the IVP $y' = y$, $y(0) = 1$.

Picard iteration starts with $y_0(t) = 1$. Then,

$$
\begin{aligned}
y_1(t) &= y_0 + \int_0^t y_0(s)\,ds && = 1 + \int_0^t 1\,ds && = 1 + t \\
y_2(t) &= y_0 + \int_0^t y_1(s)\,ds && = 1 + \int_0^t 1 + t\,ds && = 1 + t + \frac{t^2}{2} \\
y_3(t) &= y_0 + \int_0^t y_2(s)\,ds && = 1 + \int_0^t 1 + t + \frac{t^2}{2}\,ds && = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!}.
\end{aligned}
$$

So, Picard iteration leads to the power series for e^t, which we know is the solution to this IVP.