18.03 Fourier series using complex exponentials

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1 Introduction

We've learned to write Fourier series in terms of sines and cosines. We can also construct them using complex exponentials. As we've seen, there are many computations benefits to replacing $\cos(x)$ or $\sin(x)$ by e^{ix} .

For ease of notation we assume periodic functions of period 2π and work on the interval $[-\pi, \pi]$. The extension to other periods is straightforward..

2 Fourier theorem

First we'll give the Fourier theorem and then we'll motivate and prove it.

Theorem (Fourier): Suppose $f(t)$ is continuous and periodic with period 2π. Then

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}, \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt \tag{1}$$

3 Orthogonality relations

The proof of the Fourier theorem uses the orthogonality relations:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\cdot e^{-imt}\,dt = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

The proof of the orthogonality relations is a trivial integration. (The statement and proof are much easier than the analogous statements with sines and cosines.)

4 Proof of the formula for the Fourier coefficients c_n

We start proof Fourier's theorem by showing that, if $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$, then the coefficients c_n are given by the integral formulas in Equation 1.

This is a direct consequence of the orthogonality relations:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cdot e^{-in_0t}\,dt = \sum_{n=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} c_n e^{int}\cdot e^{-in_0t} = c_{n_0}.$$

The last equality follows from the orthogonality relations, which tell us that all the integrals in the sum are zero except for the term with $n = n_0$.

5 Proof every continuous, periodic function equals its Fourier series

To finish the proof of Fourier's theorem, we need to show that every continuous, periodic function equals its Fourier series. For this, see the note on Fourier completeness. Since $\cos(t)$ is a sum of complex exponentials the proof there suffices.

6 Comments

1. The *bilinear form* $\langle f, g\rangle = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\bar{g}(t)\,dt$ is an inner product on the vector space of periodic functions. The Fourier theorem and orthogonality relations show the functions $\{e^{int}\}$ form an *orthonormal basis*. They are analogous to the standard vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in three-dimensional space.

2. Also see the note on Fourier completeness for the definition of convolution and the periodic delta function.

3. The following notation is often used for the Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cdot e^{-int}\,dt.$$

With this notation the Fourier theorem says

$$f(t) = \sum_{-\infty}^{\infty} \hat{f}(n)\,e^{int}.$$

Theorem: Fourier series connect convolution and multiplication:

$$\widehat{f*g}(n) = 2\pi\,\hat{f}(n)\cdot\hat{g}(n).$$

Proof: The proof is simple algebra, but it gives some insight into the algebraic meaning of convolution.

First note that

$$e^{int} * e^{imt} = \int_{-\pi}^{\pi} e^{in(t-u)}e^{imu}\,du = e^{int}\int_{-\pi}^{\pi} e^{i(m-n)u}\,du = \begin{cases} 2\pi e^{int} & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Now

$$\begin{aligned} (f*g)(t) &= \sum_n \hat{f}(n)e^{int} * \sum_m \hat{g}(m)e^{imt} \\ &= \sum_{n,m} \hat{f}(n)\hat{g}(m)\,e^{int} * e^{imt} \qquad \text{(by the formula just above, most of these terms are 0)} \\ &= \sum_n 2\pi\hat{f}(n)\hat{g}(n)e^{int} \end{aligned}$$

This last equality shows the Fourier coefficients of $f*g$ are $2\pi\,\hat{f}(n)\hat{g}(n)$ as claimed.

Note: In the Fourier game the factors of 2π that occur throughout are sometimes put in other places. For example, some authors define

$$\hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)e^{-int}\,dt$$

and then

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum \hat{f}(n)e^{int}$$