ES.1803 Lab Exercise: Fourier Sound Analysis Solutions (30 points)

In these problems we will use the 'FourierSound' and 'BeatsWithSound' applets to explore the connection between Fourier series and musical sounds. The applets give you a visual and aural connection to the mathematics.

First open the 'FourierSound' applet:

https://web.mit.edu/jorloff/www/OCW-ES1803/fourierSound-jmo.html

For sound quality, headphones are generally better than your laptop's speakers. This is especially true for low frequency sine waves, which most laptop speakers have trouble with.

Please start with the volume low and increase it slowly until you can hear the sounds comfortably. Loud pure sine waves are more damaging to your ears than other loud sounds.

Play around with the applet –the help (click the button on the upper right of the window) will provide some guidance. Understand what the coefficients are telling you. They can be set to both rectangular and polar form:

$$a_n \cos\left(n\frac{\pi}{L}t\right) + b_n \sin\left(n\frac{\pi}{L}t\right) = A_n \cos\left(n\frac{\pi}{L}t - \phi_n\right).$$

Problem A (9: 3,3,3)*In this part we will look at the harmonics of the triangle wave. Refresh the browser page to reset the applet. Then choose triangle wave from the sounds dropdown menu and set the frequency to 256 hz. Set the zoom to show about 20 ms of wave form. Look at and listen to the sound.*

(i) *This is an even triangle wave with amplitude 1. Write down the general expression for the Fourier coefficients. (You can start with one of our known Fourier series.) Give the decimal expression for the DC term and the next 7 pairs of Fourier coefficients.*

Verify that the applet and theory agree.

Solution: Let tri(t) be the period 2π amplitude π. We know:

$$\text{tri}(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\,odd} \frac{\cos nt}{n^2}.$$

Scaling by $1/\pi$ to make it amplitute 1, we get the function: $\frac{1}{\pi}\text{tri}(t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n\,odd}\frac{\cos nt}{n^2}$.

Call the applet wave $f(t)$. It has fundamental frequency 200 hz = 400π radians/sec = 0.4π radians/ms. So,

$$f(t) = \frac{1}{\pi}\text{tri}(0.4\pi t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n\,odd}\frac{\cos(400\pi n t)}{n^2}.$$

(Note: scaling the frequency does not change the Fourier coefficients.) Using a calculator, for the applet triangle wave, we get:

n	0	1	2	3	4	5	6	7
a_n	0.5	-0.40528	0	-0.04503	0	-0.01621	0	-0.00827
b_n		0	0	0	0	0	0	0

The pairs (a_n, b_n) given by the applet agree with those we computed.

(ii) *Set the applet to show both the original and reconstructed waveforms. How closely do the graphs agree? Play the 'original and then the 'reconstructed' sounds. How do they compare?*

Now use the checkboxes to turn off the harmonics from $n = 6$ to $n = 19$. (You can do this quickly by shift-clicking on the $n = 6$ checkbox.) How well do the original and reconstructed graphs agree in this case? Play both sounds. How do they compare.

Solution: With all the harmonics, the graphs are essentially identical, as are the sounds.

Using only the first 5 harmonics, the graphs are close, but not identical. Especially around the corner, the reconstructed graph is more rounded. On my system, the sounds are very close. The original is a little buzzier and brighter due to the presence of the higher pitched harmonics.

(iii) *Zoom in so that the graph shows about 8 ms of the waveform. You should see two full cycles of the 256 hz waveform.*

Now uncheck the coefficients for all harmonics except the fundamental, i.e., turn off the coefficients 2-19. (You can turn off harmonics 2-19 by shift-clicking on the checkbox for coefficient pair 2.)

Play separately both the 'original sound' and the 'reconstructed' one with the harmonics turned off. How do the sounds compare?

Why is the reconstructed sound not as loud?

Add in the harmonics one at a time –start by turning on 2 then 3 etc. (After turning on a harmonic, give it a few seconds till you stop hearing two distinct pitches – your brain (or at least my brain) takes a while to do the Fourier synthesis!) After turning on a harmonic, listen to the original and reconstructed sounds. What happens to the graphs and the sounds as the harmonics are added? Does anything change when you add in the even numbered harmonics?

Solution: The original sound is much 'buzzier' than the first harmonic alone. The original wave is louder because it is the sum of the first harmonic and *all* the later harmonics, which add volume.

As the harmonics are turned on the graph goes from the gradually undulating sine curve to the more angular triangle wave. Likewise, the sound goes from the pure sine wave to the buzzy triangle wave. The even harmonics have zero amplitude so they don't change the graph or sound.

Problem B (9: 3,3,3)*In this part we will look at how the DC term and the phase affect the sound. Use the sounds menu to make a square wave. Set the frequency to 200 hz. Turn on (check) the coefficients for all the harmonics. (Again, shift clicking can speed this up.)*

(i) *Make sure both gains are set to 0 db. Play the reconstructed wave form. Leave the sound playing and change the DC term (the a_0 entry) by slowly moving the slider for that term. How does the reconstructed graph change? How does the reconstructed sound change?*

Solution: The reconstructed graph is shifted up. The sound doesn't change until it starts 'clipping' at the top rail. At this point the volume decreases because the amplitude is decreasing. It also become buzzier, indicating that the relative amplitude of the higher harmonics has increased.

(ii) *Turn off harmonics 2-19 (leave $n = 0$ and $n = 1$ checked). Now repeat Part (i), by playing the reconstructed wave and slowly increasing the DC term. How does the sound change?*

Solution: The sound doesn't change until it starts 'clipping' at the top rail. At this point it starts to sound buzzy, like a square wave.

(iii) *Turn on all the harmonics and reset the DC term to 0.0. Select 'amplitude-phase'. Now the coefficent pairs represent A_n and ϕ_n. (See the formula above Problem A.) Give the formulas for A_n and ϕ_n in terms of a_n and b_n. (Do this for a general Fourier series, not just for the square wave.)*

Solution: This is standard amplitude phase form:

$$A_n = |a_n + ib_n| = \sqrt{a_n^2 + b_n^2} \text{ and } \phi_n = \operatorname{Arg}(a_n + ib_n) = \tan^{-1}\left(\frac{b_n}{a_n}\right), \text{ in the correct quadrant.}$$

(iv) *Reset the 200 hz. square wave. Set both gains to -3.0 db. (This will prevent clipping when we change the phase). Select amplitude-phase for the coefficient pairs. Play the reconstructed sound. Leave the sound playing and randomly adjust the phase (ϕ_n) sliders. How does the reconstructed graph change? How does the reconstructed sound change?*

Solution: The graph is completely different, but the sound doesn't change. Our ears do not hear phase differences between the Fourier components.

Problem C (12: 3,3,3,3)*In this question we'll look at the phenomenon of beats.*

Open the Beats applet and get familiar with it.

`https://web.mit.edu/jorloff/www/OCW-ES1803/beatsWithSound.html`

The help will explain all the controls. Here are a few pointers.

1. The 'mixed wave' is the average of two sine waves of different frequencies. The frequencies are controlled by the sliders marked 'freq 1' and 'freq 2'.

2. The little triangles at the end of the frequency sliders can be used to fine tune the frequency. Clicking on the triangles raises or lowers the frequency by 0.1 hz. Shift-clicking increments it by 1 hz. (Option/alt clicking increments it by 10 hz.)

(i) *Configure the applet by:*

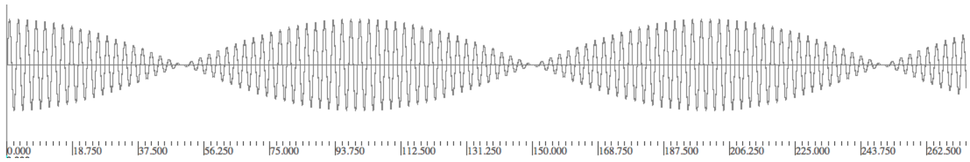
1. Set Frequency 1 to 390 hz. and Frequency 2 to 400 hz.
2. Set the zoom so that the graph window shows about 250 ms of waveform.
3. Set the applet to show only the mix waveform.

Play in order: Sound 1; Sound 2; both together.

When both sounds are played together you should hear a fairly rapid set of 'beats'. (If not go back and try again.)

Sketch what you see and explain why this explains the beats you heard.

Solution: The sound has a constant pitch, but the volume fades in and out. The picture shows a high frequency sine wave with its amplitude oscillating inside an envelope shaped like a low frequency sine wave. On the graph we can measure one fade in/fadeout cycle takes 100 ms, i.e., there are 10 cycles per second.



(ii) *The arithmetic key to beats is the trig identity*

$$\frac{\sin(at)+\sin(bt)}{2} = \cos\left(\frac{(a-b)t}{2}\right)\cdot\sin\left(\frac{(a+b)t}{2}\right).$$

For a and b close together, $(a-b)/2$ is small and $(a+b)/2$ is almost the same as a or b. This says the sum of the two sine waves looks like a high frequency sine wave ($\sin((a+b)t/2)$) with slowly changing amplitude ($\cos((a-b)t/2)$).

Work this out in the case $a = 390\cdot 2\pi$ and $b = 400\cdot 2\pi$ and give the frequency of the beats. Does this match what you see on the screen?

Solution: The applet shows the average of the two sine waves, i.e., $\dfrac{\sin(390\cdot 2\pi\, t)+\sin(400\cdot 2\pi\, t)}{2}$ (with t in seconds). The trig formula given says the mixed wave is

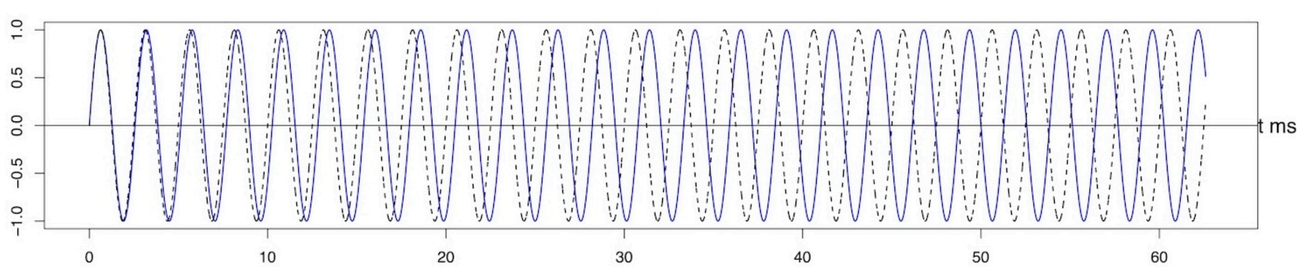
$$\frac{\sin(390\cdot 2\pi\, t)+\sin(400\cdot 2\pi\, t)}{2} = \cos(5\cdot 2\pi\, t)\sin(395\cdot 2\pi\, t).$$

Since $\cos(5\cdot 2\pi\, t)$ has frequency 5 cycles/second, it goes to zero 10 times per second. So, the sound fades 10 times per second, i.e., the beats have a frequency of 10 cycles/second. This matches what you hear and the graph of the mixed sound. (You can easily measure the beat period from the graph –it is 100 ms.)

(iii) *Another explanation for beats comes by looking at the graphs of the individual sounds.*

Set the applet as in Part (ii). Except: show sound 1 and sound 2 (but not mix), and zoom so that the graph window shows about 50 ms of waveform. Sketch what you see and use it to explain the beats.

Solution: The time axis is in milliseconds. At time $t = 0$ the two graphs are fully in phase and so the sounds add constructively. The slightly different frequencies causes them to drift apart. At 50 ms they are fully out of phase and the sounds add destructively. At 100 ms they will be back fully in phase. The whole beat cycle takes 100 ms. That is, it happens 10 times per second.



Solid blue: $\sin(390\cdot 2\pi\cdot t)$, dashed black: $\sin(400\cdot 2\pi\cdot t)$ (t in seconds)

(iv) *Set the applet so that 'show mix' is on and 'show 1' and 'show 2' are off. Set the frequencies to 399 hz. and 400 hz. You might want to zoom out so about 3000 ms (3 seconds) are shown to see the beats on the screen. How does the quality of the mixed sound* *differ from that in Part (i)? What if you set the frequencies to 399.8 hz (use the fine tuning arrows) and 400 hz?*

Solution: The beats have a much lower frequency fading in and out once per second. In terms of the analysis in Part (ii) the slowly varying amplitude is $\cos(0.5 \cdot 2\pi\, t)$. This has period 2 seconds, so it goes to zero once per second, which is the frequency of the beats.

At 399.8 hz and 400 hz. The slowly varying amplitude has period 10 seconds, so the beats have period 5 seconds.

(v) *Now set the frequencies to 310 hz. and 400 hz. When you play both sounds together you shouldn't hear any beats, but instead two distinct tones like a train whistle.*

There is no question in Part (v).

The following are only for fun.

Problem D (for fun)
(i) *In the Fourier sound applet use the scale menu to connect notes and frequencies. Notice that an octave is a doubling of the frequency. That is, doubling the frequency results in a different pitch but the same note.*

(ii) *In the Fourier Sound applet see that the different waveforms of the same frequency give the same note.*

(iii) *In the Fourier Sound applet listen to a 256 hz. rectified sine wave. (You may need to use the shift arrow to get the slider to 256.) Now remove the fundamental frequency (turn off coefficient pair 1). To make the sounds have roughly equal loudness set the original wave gain to about* -2.0 *db and the reconstructed wave gain to* 2.0 *db..*

Toggle between playing the original and reconstructed wave. Relative to the original pitch, what is the pitch of the reconstructed sound without the fundamental? When you toggle, give each sound a moment for your ears to adjust to the change.

Solution: This is dependent on both ears and audio systems: About 50% of people will hear the same pitch. The reason is that they have the same fundamental frequency, i.e, the higher harmonic frequencies are multiples of 25k hz. Other ears, will hear the separate frequencies as a tone one octave up from the original and a collection of higher tones.

(iv) *In the Fourier Sound applet, zoom in on the discontinuous sawtooth wave and see Gibbs' phenomenon occurring at the jumps. Turn off the harmonics starting from coefficient 19 to see the overshoot move away from the jump. Do the same thing with the square wave.*

End of Fourier Sound solutions.