ES.1803 Part I Solutions

Topic 1. Introduction to DEs; modeling; separable equations

Solutions

1.1. **Solution:** $x'(t) = kx(P - x)$, where k is the constant of proportionality.

1.2. **Solution:** Let $T(t)$ be the temperature of the root beer. We'll model this using Newton's law of cooling:

$$T'(t) = -k(T - E); \quad T(0) = 20,$$

where t is time in minutes, k is the cooling rate in units of 1/min and $E = 0$ is the temperature of the environment.

Since $E = 0$, we can write this more simply as $T' + kT = 0, \quad T(0) = 20$.

This is the model for exponential decay. It's solution is $T(t) = 20e^{-kt}$.

We can find k using the temperature at $t = 30$ min.: $T(30) = 20e^{-30k} = 10$.

So, $-30k = \ln(0.5)$, which gives $k = -\ln(0.5)/30 \approx 0.0231$.

Now we can solve for the time t when $T(t) = 4$: $20e^{-kt} = 4$ or $-kt = \ln(0.2)$.
So, $\boxed{t = -\ln(0.2)/k \approx 70 \text{ minutes.}}$

1.3. **Solution:** Separating variables gives $y^2\,dy = \dfrac{dx}{\ln(x)}$. If we want a definite integral with the variable x in the limit we need to use a dummy variable. Integrating with u going from 2 to x gives

$$\int_{y(2)}^{y(x)} y^2\,dy = \int_2^x \frac{du}{\ln(u)} \Rightarrow \left.\frac{y^3}{3}\right|_{y(2)}^{y(x)} = \int_2^x \frac{du}{\ln(u)}.$$

That is, $\dfrac{y(x)^3}{3} - \dfrac{y(2)^3}{3} = \displaystyle\int_2^x \frac{du}{\ln(u)}$.

Now use the initial condition $y(2) = 0$ and solve for y:

$$\boxed{y(x) = \left[3\int_2^x \frac{du}{\ln(u)}\right]^{1/3}.}$$

1.4. **Solution:** Separating variables gives $\dfrac{y\,dy}{y+1} = x\,dx$.

Integrate (noting that $\dfrac{y}{y+1} = 1 - \dfrac{1}{y+1}$): $y - \ln(y+1) = \dfrac{x^2}{2} + c$.

Use $y(2) = 0$ to find $c = -2$. So, $\boxed{y - \ln(y+1) = \frac{x^2}{2} - 2.}$.

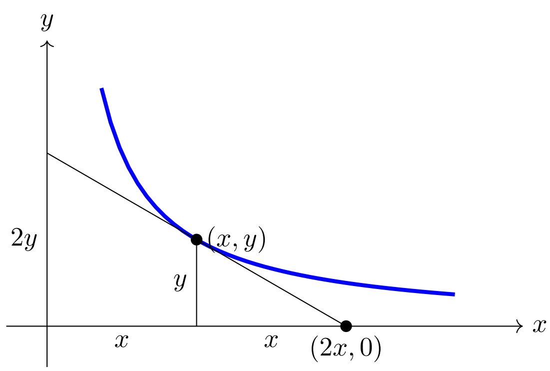
1.5. Solution: Separate variables: $\frac{dv}{\sqrt{1-v^2}} = \frac{dx}{x}$.

Integrating gives: $\sin^{-1}(v) = \ln|x| + c$ or $\boxed{v = \sin(\ln|x| + c).}$

1.6. Solution: The figure shows the piece of the tangent bisected by the point (x, y) on the curve. Geometrically, we see the tangent line contains the points (x, y) and $(2x, 0)$. So the slope of this line is $\frac{0-y}{2x-x} = -\frac{y}{x}$. Since the slope is also the derivative dy/dx, we have the ODE:

$$\frac{dy}{dx} = \frac{-y}{x}.$$

This differential equation is separable and is easily solved: $\boxed{y = C/x.}$



Topic 2. Linear systems: input-response models

Solutions

2.1. Solution: Variation of parameters solution.

In standard form the equation is $y' + \frac{2}{x}y = 1$.

The homegenous equation $y' + \frac{2}{x}y = 0$ has solution $y_h(x) = e^{-\int \frac{2}{x}\,dx} = e^{-2\ln(x)} = 1/x^2$.

The variation of parameters formula is

$$y(x) = y_h(x)\left(\int q(x)/y_h(x)\,dx + c\right).$$

In this case this becomes

$$\boxed{y(x) \;=\; \frac{1}{x^2}\left(\int x^2\,dx + c\right) \;=\; \frac{x}{3} + \frac{c}{x^2}}.$$

2.2. Solution: Using the variation of parameters formula we have: $x_h(t) = e^{-at}$ and

$$x(t) = e^{-at}\left[\int \frac{r(t)}{e^{-at}}\,dt + c\right] = \frac{\int r(t)e^{at}\,dt}{e^{at}}$$

Since $a > 0$, the denominator in the last expression above goes to infinity.

First, assume the numerator also goes to infinity. Then we can use L'Hospital's rule to find the limit. (The derivative of the numerator is immediate using the fundamental theorem of calculus.)

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{\int r(t)e^{at}\,dt}{e^{at}} = \lim_{t\to\infty} \frac{r(t)e^{at}}{ae^{at}} = \lim_{t\to\infty} \frac{r(t)}{a} = 0.$$

Second, assume the numerator does not go to infinity, then clearly the ratio goes to 0.

2.3. Solution: Newton's law of cooling says

$$\frac{dT}{dt} = -k(T - 20),$$

where k is the constant of proportionality. Solving (using separation of variables or variation of parameters) we have

$$T(t) = ae^{-kt} + 20.$$

Using $T(0) = 100$, we get $T(0) = 100 = a + 20$, so $a = 80$ and $T(t) = 80e^{-kt} + 20$.

Similarly, using $T(5) = 80$, we get $T(5) = 80 = 80e^{-5k} + 20$. Solving for k, we find $k = -\frac{\ln(3/4)}{5} > 0$.

Thus we have $T(t) = 80e^{\ln(3/4)t/5} + 20$.

When $T = 60$, we have $60 = 80e^{\ln(3/4)t/5} + 20$. Solving for t, we find $t = \dfrac{5\ln(2)}{\ln(4/3)} \approx 12$ minutes.

Topic 3. Input-response models continued

Solutions

3.1. Solution: (a) We use $x' =$ rate of salt in - rate of salt out.

So, $x'(t) = -5x/100$, $x(0) = 30$. Solving we get: $\boxed{x(t) = 30e^{-t/20}}$.

(b) We use $y' =$ rate of salt in - rate of salt out.

So, $y'(t) = 5x/100 - 5y/100$, $y(0) = 15$. This is first-order linear DE. Rearranging, we get

$$y' + \frac{y}{20} = \frac{x}{20} = \frac{3}{2}e^{-t/20}.$$

We solve with the variation of parameters formula: The homogeneous solutions is $y_h(t) = e^{-t/20}$,

$$\begin{aligned} y(t) &= y_h(t) \int \frac{f(t)}{y_h(t)}\, dt = e^{-t/20} \int \frac{3}{2} e^{-t/20} e^{t/20}\, dt \\ &= e^{-t/20} \int \frac{3}{2}\, dt = e^{-t/20} \left(\frac{3}{2}\, t + C \right) \\ &= \frac{3}{2}\, te^{-t/20} + Ce^{-t/20}. \end{aligned}$$

We find C by using the initial condition: $y(0) = C = 15$, so $C = 15$.

Solution: $\boxed{y(t) = \frac{3}{2}\, te^{-t/20} + 15e^{-t/20}}$.

Topic 4. Complex numbers and exponentials

Solutions

4.1. Solution: Method 1. $\dfrac{1-i}{1+i} = \dfrac{\sqrt{2}e^{-i\pi/4}}{\sqrt{2}e^{i\pi/4}} = e^{-i\pi/2} = -i.$

Method 2. $\dfrac{1-i}{1+i} = \dfrac{1-i}{1+i} \cdot \dfrac{1-i}{1-i} = \dfrac{-2i}{2} = -i.$

4.2. (a) Solution: Method 1: $(1-i)^4 = (\sqrt{2}e^{-i\pi/4})^4 = 4e^{-i\pi} = -4.$

Method 2: $(1-i)^4 = 1^4 - 4 \cdot 1^3 i + 6 \cdot 1^2 i^2 - 4 \cdot 1 i^3 + i^4 = 1 - 4i - 6 + 4i + 1 = -4$

(b) Solution: Method 1: $(1+i\sqrt{3})^3 = (2e^{i\pi/3})^3 = 8e^{i\pi} = -8.$

Method 2: $(1+i\sqrt{3})^3 = 1^3 + 3 \cdot 1^2(i\sqrt{3}) + 3 \cdot 1(i\sqrt{3})^2 + (i\sqrt{3})^3 = 1 + 3i\sqrt{3} - 9 - i3\sqrt{3} = -8$

4.3. Solution: Using Euler's formula we can write $e^{3i\theta}$ two ways.

$$e^{3i\theta} = \cos(3\theta) + i\sin(3\theta)$$
$$e^{3i\theta} = (e^{i\theta})^3 = (\cos(\theta) + i\sin(\theta))^3 = \cos(\theta)^3 + i3\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)$$

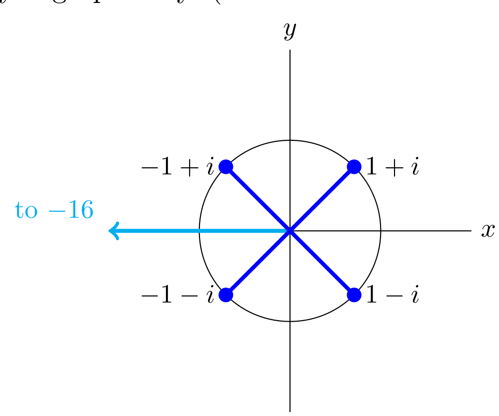
Equating the real and imaginary parts of each expression we have

$$\cos(3\theta) = \cos(\theta)^3 - 3\cos(\theta)\sin^2(\theta), \qquad \sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)$$

4.4. (a) Solution: $x^4 = -16 = 16e^{i\pi + i2\pi k}$, $k = 0, 1, 2, \ldots.$ So,

$$\begin{aligned} x &= 2e^{i\pi/4 + i2\pi k/4} = 2e^{i\pi/4},\ 2e^{i3\pi/4},\ 2e^{i5\pi/4},\ 2e^{i7\pi/4} \\ &= \boxed{\sqrt{2}(1+i),\ \sqrt{2}(-1+i),\ \sqrt{2}(-1-i),\ \sqrt{2}(1-i) = \sqrt{2}(\pm 1 \pm i)}. \end{aligned}$$

Here are the roots displayed graphically. (-16 doesn't fit on the graph)



(b) Solution: This is quadratic in x^2, so using the quadratic formula

$$x^2 = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm i\sqrt{3} = 2e^{i2\pi/3},\ 2e^{i4\pi/3}.$$

Thus,

$$\boxed{x = \pm\sqrt{2}e^{i\pi/3},\ \pm\sqrt{2}e^{i2\pi/3} = \pm\sqrt{2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), \pm\sqrt{2}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}$$

4.5. **Solution:** Let $I_c = \displaystyle\int e^{3x}(\cos(4x) + i\sin(4x))\,dx = \int e^{(3+4i)x}\,dx$. It is clear that $I = \text{Im}(I_c)$.

Computing: $I_c = \dfrac{e^{(3+4i)x}}{3+4i}$

Using polar form: $3 + 4i = 5e^{i\phi}$, where $\phi = \text{Arg}(3+4i) = \tan^{-1}(4/3)$ in the first quadrant. So,

$$I_c = e^{3x}\frac{e^{4ix}}{5e^{i\phi}} = \frac{e^{3x}}{5}e^{i(4x-\phi)}.$$

Thus, $\boxed{I = \text{Im}(I_c) = \dfrac{e^{3x}}{5}\sin(4x - \phi).}$

Topic 5. Homogeneous, linear, constant coefficient DEs

Solutions

5.1. (a) Solution: Characteristic equation: $r^2 - 3r + 2 = 0$.

Roots: $r = 1,\ 2$.

Modal solutions: $y_1(t) = e^t,\ \ y_2(t) = e^{2t}$.

General solution by superposition: $y(t) = c_1 y_1 + c_2 y_2 = c_1 e^t + c_2 e^{2t}$.

(b) Solution: Characteristic equation: $r^2 + 2r + 2 = 0$.

Roots: $r = -1 \pm i$.

Two solutions: $y_1(t) = e^{-t}\cos(t),\ \ y_2(t) = e^{-t}\sin(t)$.

General *real-valued* solution by superposition:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) = Ae^{-t}\cos(t - \phi).$$

(c) Solution: Characteristic equation: $r^2 - 2r + 5 = 0$.

Roots: $r = 1 \pm 2i$.

Two solutions: $y_1(t) = e^t\cos(2t),\ \ y_2(t) = e^t\sin(2t)$.

General *real-valued* solution by superposition:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^t\cos(2t) + c_2 e^t\sin(2t) = Ae^t\cos(2t - \phi).$$

For the initial condition it's easiest to use the rectangular form of the solution.

$$\begin{aligned} y(0) &= 1 = c_1 \\ y'(0) &= -1 = c_1 + 2c_2 \ \Rightarrow\ c_2 = -1. \end{aligned}$$

So, $\boxed{y(t) = e^t\cos(2t) - e^t\sin(2t).}$

5.2. (a) Solution: Characteristic equation: $r^6 - 1 = 0$.

Roots: $r = e^{i2\pi n/6}$, $n = 0, 1, 2, 3, 4, 5$. So, $r = \pm 1,\ \dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2},\ -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

Modal solutions: $y_1(t) = e^t,\ \ y_2(t) = e^{-t},\ \ y_3 = e^{t/2}\cos(\sqrt{3}t/2),\ \ y_4 = e^{t/2}\sin(\sqrt{3}t/2)$,
$y_5 = e^{-t/2}\cos(\sqrt{3}t/2),\ \ y_6 = e^{-t/2}\sin(\sqrt{3}t/2)$

General solution by superposition: $y(t) = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4 + c_5 y_5 + c_6 y_6$.

(b) Solution: Characteristic equation: $r^4 + 16 = 0$.
Roots: $r = 2e^{i\pi/4 + i2\pi n/4}$, $n = 0, 1, 2, 3$. So, $r = \pm\sqrt{2}\ \pm i\sqrt{2}$.

Modal solutions: $y_1(t) = e^{\sqrt{2}t}\cos(\sqrt{2}t),\ \ y_2(t) = e^{\sqrt{2}t}\sin(\sqrt{2}t),\ \ y_3 = e^{-\sqrt{2}t}\cos(\sqrt{2}t)$,
$y_4 = e^{-\sqrt{2}t}\sin(\sqrt{2}t)$

General solution by superposition: $y(t) = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$

Topic 6. Operators, ERF and SRF

Solutions

Note: the Exponential Response Formula is also called the Exponential Input Theorem.

6.1. (a) Solution: Particular solution (using ERF): The characteristic polyomial is $P(r) = r^2 + 6r + 12$, so

$$y_p(t) = \frac{e^{2t}}{P(2)} = \frac{e^{2t}}{28}.$$

Homogeneous solution: $P(r) = 0 \Rightarrow$ characteristic roots are $r = -3 \pm \sqrt{3}i$.

So, $y_h(t) = c_1 e^{-3t}\cos(\sqrt{3}t) + c_2 e^{-3t}\sin(\sqrt{3}t)$. The general solution is

$$y(t) = y_p + y_h = \frac{e^{2t}}{28} + c_1 e^{-3t}\cos(\sqrt{3}t) + c_2 e^{-3t}\sin(\sqrt{3}t).$$

(b) Solution: More quickly than in Part (a): $P(r) = r^2+4r+12$. So, $P(-2) = 4-8+12 = 8 \Rightarrow y_p(t) = \frac{e^{-2t}}{8}$.

6.2. (a) Solution: Homogenous solution: (from Problem 1) $y_h(t) = c_1 e^{-3t} + c_2 e^{-4t}$

Particular solution: Complexify: $z'' + 7z' + 12z = e^{2it}$, $y = \text{Re}(z)$.

Exponention Response Formula: $P(2i) = 8 + 14i = 2\sqrt{65}e^{i\phi}$, where

$$\boxed{\phi = \text{Arg}(8 + 14i) = \tan^{-1}(7/4) \text{ in Q1.}}$$

So, $z_p(t) = \frac{e^{2it}}{P(2i)} = \frac{e^{2it}}{2\sqrt{65}e^{i\phi}} = \frac{e^{i(2t-\phi)}}{2\sqrt{65}}$. Therefore, $y_p(t) = \text{Re}(z_p) = \dfrac{\cos(2t-\phi)}{2\sqrt{65}}$.

The general solution is $\boxed{y(t) = y_p + y_h = \dfrac{\cos(2t-\phi)}{2\sqrt{65}} + c_1 e^{-3t} + c_2 e^{-4t}.}$

(b) Solution: Use z_p from Part (a): $y_p(t) = \text{Im}(z_p) = \boxed{\dfrac{\sin(2t-\phi)}{2\sqrt{65}}}$.

(c) Solution: Complexify: $P(D)z = e^{(2+3i)t}$, $y = \text{Re}(z)$.

Use ERF: $P(2+3i) = 21+33i = 3\sqrt{170}e^{i\phi}$, where $\boxed{\phi = \text{Arg}(21 + 33i) = \tan^{-1}(11/7) \text{ in Q1.}}$

$$z_p(t) = \frac{e^{(2+3i)t}}{3\sqrt{170}e^{i\phi}} = \frac{e^{2t}e^{i(3t-\phi)}}{3\sqrt{170}} \Rightarrow \boxed{y_p(t) = \text{Re}(z_p) = \frac{e^{2t}\cos(3t-\phi)}{3\sqrt{170}}.}$$

Homogenous solution: (from previous problems) $\boxed{y_h(t) = c_1 e^{-3t} + c_2 e^{-4t}}$.

General solution is $y(t) = y_p + y_h = \dfrac{e^{2t}\cos(3t-\phi)}{3\sqrt{170}} + c_1 e^{-3t} + c_2 e^{-4t}$.

6.3. Solution: Try the ERF: $P(-4) = 0$. So we must use the extended ERF: $y_p(t) = \dfrac{te^{-4t}}{P'(-4)}$.

$P(r) = r^2 + 7r + 12 \Rightarrow P'(r) = 2r + 7 \Rightarrow P'(-4) = -1 \Rightarrow \boxed{y_p(t) = -te^{-4t}.}$

6.4. (a) Solution: Try the sinusoidal response formula: We have $P(r) = r^2 + 9$, so $P(i) = 8$.

Thus, $|P(i)| = 8$ and $\text{Arg}(P(i)) = 0 \quad \Rightarrow \boxed{y_p(t) = \dfrac{\cos(t)}{8}}$.

(b) Solution: We have $P(r) = r^2 + 9$. Since $P(3i) = 0$, we must use the extended version of the SRF.

$$y_p(t) = \frac{t\cos(3t - \phi)}{|P'(3i)|}, \quad \text{where } \phi = \text{Arg}(P'(3i)).$$

$P'(r) = 2r$. So, $P'(3i) = 6i \quad \Rightarrow |P'(3i)| = 6,\ \phi = \text{Arg}(P'(3i)) = \pi/2$.

Thus, $\boxed{y_p(t) = \dfrac{t\cos(3t - \pi/2)}{6} = \dfrac{t\sin(3t)}{6}}$.

Note: We also could have done this by complexifying and using the extended ERF.

6.5. Solution: We have $P(r) = r^4 + 2r^2 + 4$.

First we find the homogeneous solution. The characteristic polynomial, $r^4 + 2r^2 + 4 = 0$, is a quadratic in r^2. That is, if $u = r^2$, then $u^2 + 2u + 4 = 0$ so we have roots

$$r^2 = -1 \pm i\sqrt{3} = 2e^{i2\pi/3}, 2e^{-i2\pi/3}.$$

Taking the two square roots for each of these we have

$$r = \sqrt{2}e^{i\pi/3},\ -\sqrt{2}e^{i\pi/3},\ \sqrt{2}e^{-i\pi/3},\ -\sqrt{2}e^{-i\pi/3} = \frac{\sqrt{2}}{2} \pm i\frac{\sqrt{6}}{2},\ -\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{6}}{2}.$$

So the general real-valued homogeneous solution is

$$\boxed{y_h(t) = c_1 e^{\sqrt{2}t/2}\cos(\sqrt{6}t/2) + c_2 e^{\sqrt{2}t/2}\sin(\sqrt{6}t/2) + c_3 e^{-\sqrt{2}t/2}\cos(\sqrt{6}t/2) + c_4 e^{-\sqrt{2}t/2}\sin(\sqrt{6}t/2)}.$$

To find the particular solution, we use the sinusoidal response formula: $y_p(t) = \dfrac{\cos(3t - \phi)}{|P(3i)|}$.

$P(3i) = 81 - 18 + 4 = 67$, so $|P(3i)| = 67$ and $\phi = \text{Arg}(P(3i)) = 0$. Thus we have $\boxed{y_p(t) = \dfrac{\cos(3t)}{67}}$.

The general real-valued solution is $\boxed{y(t) = y_p(t) + y_h(t).}$

Topic 7. Inhomegenous DEs; UC methods; theory

Solutions

7.1. (a) Solution: The input is a polynomial of degree 1, so we try a solution of the same degree: $y_p(t) = At + B$. Computing derivatives and substituting this into the DE we get:

$$\begin{aligned} y_p &= At + B \\ y_p' &= A \\ y_p'' &= 0 \\ y_p'' - y_p' + 3y_p &= 3At + (-A + 3B) = 3t + 5. \end{aligned}$$

Equating coefficients we get:

$$\begin{array}{lll} \text{Coefficients of } t: & 3A & = 3 \\ \text{Coefficients of } 1: & -A + 3B & = 5 \end{array}$$

Solving these equations we get: $A = 1$, $B = 2$. Our answer is $\boxed{y_p(t) = t + 2.}$

(b) Solution: The input is a polynomial of degree 2, so we try a solution of the same degree: $y_p(t) = At^2 + Bt + C$. Computing derivatives and substituting this into the DE we get:

$$\begin{aligned} y_p &= At^2 + Bt + C \\ y_p' &= 2At + B \\ y_p'' &= 2A \\ y_p'' + 8y_p' + 7y_p &= 7At^2 + (16A + 7B)t + (2A + 8B + 7C) = t^2 \end{aligned}$$

Equating coefficients we get:

$$\begin{array}{lll} \text{Coefficients of } t^2: & 7A & = 1 \\ \text{Coefficients of } t: & 16A + 7B & = 0 \\ \text{Coefficients of } 1: & 2A + 8B + 7C & = 0 \end{array}$$

Solving these equations we get: $A = 1/7$, $B = -16/49$, $C = 114/7^3$. Our answer is $\boxed{y_p(t) = t^2/7 - 16t/49 + 114/7^3.}$

(c) Solution: Since the input is constant (degree 0) we try a constant solution $y_p(t) = A$. This is easy to substitute into the DE. We get $2A = 21$, so $\boxed{y_p(t) = 21/2.}$

7.2. (a) Solution: Differentiating $y = t^2$ we have: $y = t^2$; $y' = 2t$; $y'' = 2$. There are many ways to combine these to get 0. Here is one

$$y'' + \frac{1}{t}y' - \frac{4}{t^2}y = 0.$$

(b) Solution: The Existence and Uniqueness Theorem requires the coefficients to be continuous. In our answer to Part (a), neither $p(t) = 1/t$ or $q(t) = -4/t^2$ is continuous at 0.

Topic 8. Applications: stability

Solutions

8.1. **Solution:** The characteristic equation is $r^2 + br + 4 = 0$. This has roots

$$r = \frac{-b \pm \sqrt{b^2 - 16}}{2}.$$

(a) The equation has oscillatory solutions if the characteristic roots are complex (more precisely: have nonzero imaginary part), This happens when $b^2 - 16 < 0$, i.e., $\boxed{|b| < 4.}$

(b) The solutions are damped oscillations if the characteristic roots are complex with negative real part. Looking at the formula for the roots, this happens when $b^2 - 16 < 0$ and $b > 0$, i.e., $\boxed{0 < b < 4.}$

8.2. **Solution:** The characteristic equation is $mr^2 + br + k = 0$. So the roots are

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

Critical damping happens when the roots are repeated, i.e., when the discriminant $\boxed{b^2 - 4mk = 0.}$

8.3. **(a)** **Solution:** The DE becomes $Lq'' + \dfrac{q}{C} = 0$. The characteristic roots are $\pm\sqrt{1/LC}\, i$. So the general solution is

$$q(t) = c_1 \cos(t/\sqrt{LC}) + c_2 \sin(t/\sqrt{LC})$$

This shows q is periodic. Note, this is our standard equation for simple harmonic motion.

(b) **Solution:** Oscillating current means complex characteristic roots. The roots are

$$\frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

These are complex when $R^2 - 4L/C < 0$.

(c) **Solution:** The DE is $Li'' + i/C = \omega E_0 \cos(\omega t)$. (Here i is current not $\sqrt{-1}$.) This is an undamped oscillator with natural frequence $\omega_0 = 1/\sqrt{LC}$. The response will be large if $\omega \approx \omega_0$.

Said differently, the amplitude of the response is

$$\frac{\omega E_0}{|P(i\omega)|} = \frac{\omega E_0}{|1/C - L\omega^2|}$$

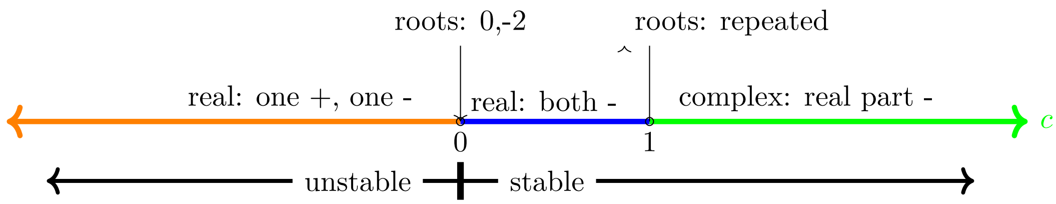
This is large when $\omega \approx 1/\sqrt{LC}$.

8.4. **(a)** **Solution:** The roots are $-1 \pm \sqrt{1-c}$.

$c < 0$: both roots are real, one is positive and one negative,
$c = 0$; roots are 0 and -2,

$0 < c < 1$; roots are real and negative,
$c = 1$; roots are repeated (-1,-1),
$c > 1$; roots are complex with negative real part.

(b) Solution:



Extra material on non-constant coefficient linear equations.

Note. In ES.1803, we used to do a little work with second-order nonconstant coefficient DEs. The following two problems are about such equations. To solve them, you will need the following formulas.

1. Wronskian of two functions: For functions $y_1(x)$ and $y_2(x)$, their Wronskian is

$$W(x) = \det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}.$$

2. Variation of parameters formula: Consider the inhomogeneous second-order linear DE and its associated homogeneous equation.

$$y'' + p(x)y' + q(x)y = f(x) \qquad \text{((I) Inhomogeneous)}$$
$$y'' + p(x)y' + q(x)y = 0 \qquad \text{((H) Homogeneous)}$$

If y_1 and y_2 are basic solutions to (H), then the solution to (I) is given by

$$y(x) = -y_1(x)\left(\int \frac{y_2(x)}{W(x)} f(x)\,dx + C_1\right) + y_2(x)\left(\int \frac{y_1(x)}{W(x)} f\,dx + C_2\right).$$

Here, $W(x)$ is the Wronskian of y_1, y_2.

8.5. (a) Solution: To use variation of parameters, we need two indendent homogeneous solutions. We find these first. Since this is constant coefficien, we could use the characteristic equation technique. Instead, we simply notice that $y'' + y = 0$ models the simple harmonic oscillator with frequency 1. Two solutions are $y_1(x) = \cos x$, $\;y_2(x) = \sin x$.

We use the variation of parameters formula given above with $f(x) = \tan x$ and the Wronskian

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x) = \cos x \cos x + \sin x \sin x = 1.$$

Thus a particular solution is

$$\begin{aligned} y(x) &= -\cos x \int \sin x \tan x \, dx + \sin x \int \cos x \tan x \, dx \\ &= -\cos x \int \frac{\sin^2 x}{\cos x} \, dx + \sin x \int \sin x \, dx \\ &= -\cos x \int \frac{1-\cos^2 x}{\cos x} \, dx - \sin x \cos x \, dx \\ &= -\cos x \int \sec x - \cos x \, dx - \sin x \cos x \\ &= -\cos x (\ln(|\sec x + \tan x|) - \sin x) - \sin x \cos x \\ &= -\cos x \ln(|\sec x + \tan x|) \end{aligned}$$

(Note: because the problem asked for *a* particular solution, we didn't include the constants C_1 and C_2 from the variation of parameters formula.)

(b) Solution: Two independent homogeneous solutions are $y_1(x) = e^x, \quad y_2(x) = e^{-3x}$. Thus, $W(x) = -4e^{-2x}$ and (keeping careful track of minus signs)

$$y(x) = \frac{e^x}{4} \int e^{-2x} \, dx - \frac{e^{-3x}}{4} \int e^{2x} \, dx = -\frac{1}{4} e^{-x}.$$

Note: the problem only calls for a particular solution, so we don't add in the general homogeneous solution.

(c) Solution: Two independent homogeneous solutions are $y_1(x) = \cos(2x), \quad y_2(x) = \sin(2x)$.

Thus, $W(x) = 2$.

$$\begin{aligned} y(x) &= -\cos(2x) \int \frac{\sin(2x)\sec^2(2x)}{2} \, dx + \sin(2x) \int \frac{\cos(2x)\sec^2(2x)}{2} \, dx \\ &= -\cos(2x) \int \sin(2x) \frac{(\cos(2x))^{-2}}{2} \, dx + \sin(2x) \int \frac{\sec(2x)}{2} \, dx \\ &= -\cos(2x) \frac{(\cos(2x))^{-1}}{4} + \sin(2x) \frac{\ln(\sec(2x) + \tan(2x))}{4} \\ &= -\frac{1}{4} + \sin(2x) \frac{\ln(\sec(2x) + \tan(2x))}{4}. \end{aligned}$$

Note: the problem only calls for a particular solution, so we don't add in the general homogeneous solution.

8.6. Solution: Wronskian $W = y_1 y_2' - y_1' y_2 = -1/x$. In order to use the variation of parameters formula, we need to write the DE in standard form as

$$y'' + \frac{1}{x} y' + \left(1 - \frac{1}{4x^2}\right) y = \frac{\cos(x)}{x^{1/2}}.$$

Thus,

$$y(x) = \frac{\sin(x)}{\sqrt{x}} \int \cos^2(x) \, dx - \frac{\cos(x)}{\sqrt{x}} \int \sin(x) \cos(x) \, dx.$$

The above answer is probably good enough. Carrying out the integration:

$$\begin{aligned} y(x) &= \frac{\sin(x)}{\sqrt{x}}\left(\frac{x}{2}+\frac{\sin(2x)}{4}\right)-\frac{\cos(x)}{\sqrt{x}}\left(\frac{\sin^2(x)}{2}\right) \\ &= \frac{\sin(x)}{\sqrt{x}}\left(\frac{x}{2}+\frac{2\sin(x)\cos(x)}{4}\right)-\frac{\cos(x)}{\sqrt{x}}\left(\frac{\sin^2(x)}{2}\right) \\ &= \boxed{\frac{\sqrt{x}\sin(x)}{2}}. \end{aligned}$$

Topic 9. Applications: frequency response

Solutions

9.1. (a) Solution: Characteristic polynomial: $P(r) = r^2 + r + 7$: $P(\omega i) = 7 - \omega^2 + i\omega$.
Particular solution (from sinusoidal response formula):

$$x_p = \frac{F_0}{|P(i\omega)|}\cos(\omega t - \phi(\omega)), \text{ where } \phi(\omega) = \text{Arg}(P(i\omega)).$$

This input amplitude is F_0 and the output amplitude is $F_0/|P(i\omega)|$, so the gain is

$$g(\omega) = \frac{1}{|p(i\omega)|} = \frac{1}{\sqrt{(7-\omega^2)^2 + \omega^2}}.$$

For graphing we analyze the term under the square root: call it $h(\omega) = (7 - \omega^2)^2 + \omega^2$.
Critical points: $h'(\omega) = -4\omega(7 - \omega^2) + 2\omega = 0 \Rightarrow \omega = 0$ or $\omega = \sqrt{13/2}$.

Evaluate at the critical points: $g(0) = 1/7$, $g(\sqrt{13/2}) \approx 0.385$

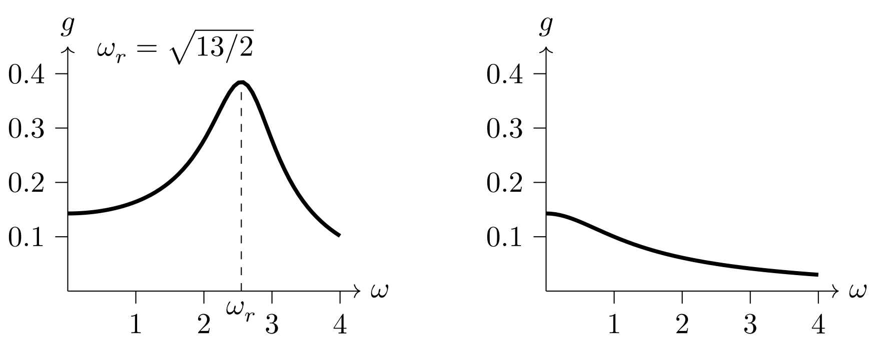
Find regions of increase and decrease by checking values of $h'(\omega)$:
On $[0, \sqrt{13/2}]$: $h'(\omega) < 0 \Rightarrow h(\omega)$ is decreasing, so $g(\omega)$ is increasing.
On $[\sqrt{13/2}, \infty]$: $h'(\omega) > 0 \Rightarrow h(\omega)$ is increasing $g(\omega)$ is decreasing.

The graph is given below. This system has a (practical) resonant frequency $= \omega_r = \sqrt{13/2}$.

(b) Solution: Characteristic polynomial: $P(r) = r^2 + 8r + 7$: $P(\omega i) = 7 - \omega^2 + i8\omega$.
Gain (similar to Part (a)): $g(\omega) = 1/|p(i\omega)| = 1/\sqrt{(7-\omega^2)^2 + 64\omega^2}$.
For graphing we analyze the term under the square root: $h(\omega) = (7 - \omega^2)^2 + 64\omega^2$.
Critical points: $h'(\omega) = -4\omega(7 - \omega^2) + 128\omega = 0 \Rightarrow \omega = 0$.

Since there are no positive critical points the graph is strictly decreasing.

Graph below.



Graphs for Problems 1a and 1b.

9.2. Solution: See solution for Topic 8 Problem 3c

9.3. (a) Solution: The second equation gives $x_1 = x_2'' + x_2$.

Now, substitute this into the left side of the first equation.

$$x_1'' + 2x_1 - x_2 = (x_2'' + x_2)'' + 2(x_2'' + x_2) - x_2 = x_2^{(4)} + 3x_2'' + x_2.$$

So the equation for x_2 is $\boxed{x_2^{(4)} + 3x_2'' + x_2 = 0.}$

(b) Solution: This is a linear, constant coefficient, homogeneous DE.

The characteristic equation is $r^4 + 3r^2 + 1 = 0$. This is a quartic, which is usually hard to solve. Happily, in this case it is a quadratic in r^2. So,

$$r^2 = \frac{-3 \pm \sqrt{5}}{2}$$

Both of these are real and negative. They are just some decimals, let's call them $-a^2$ and $-b^2$. So the characteristic roots are $r = \pm a\,i;\ \pm b\,i$. Using these roots, we find the general solution to the DE is

$$\boxed{x_2(t) = c_1 \cos(at) + c_2 \sin(at) + c_3 \cos(bt) + c_4 \sin(bt).}$$

Using a calculator, we find $\boxed{a \approx 0.618034; \quad b \approx 1.618034}$

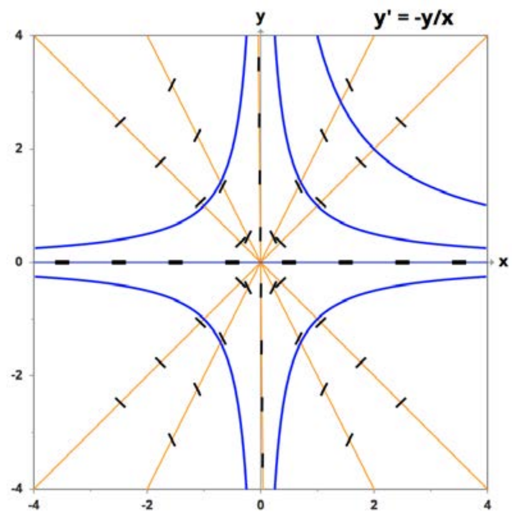
Topic 10. Direction fields, integral curves, existence of solutions

Solutions

10.1. (a) Solution: The isocline for slope m is $-y/x = m$, i.e., $y = -mx$. These are lines of slope $-m$ through the origin.

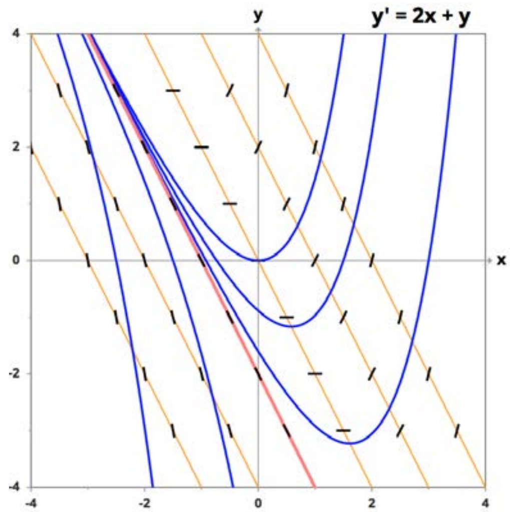
Note: it is a little confusing because there are two slopes discussed. The isocline for slope m is some curve and each slope field element drawn on it has slope m. In this case, coincidentally, the isoclines are themselves lines, so we can also talk of the slope of these lines.

The figure shows isoclines with $m = \pm 1, \pm 2, 100$. ($m = 100$ looks like $m = \infty$. It also shows sketches of a number of integral curves.



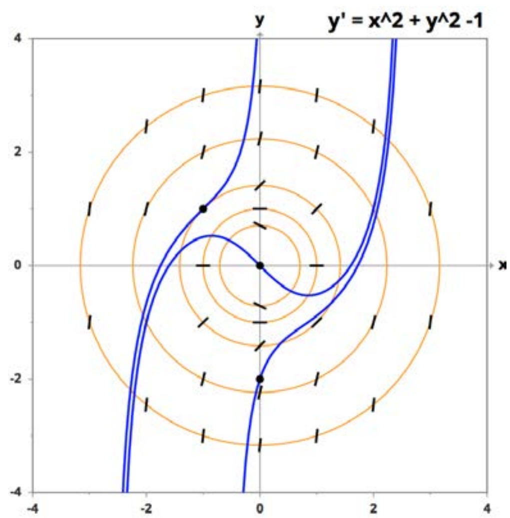
This equation is separable. The exact solution is $y = C/x$.

(b) Solution: The isoclines for slope m are the lines $2x + y = m$. These are lines with slope -2 and y-intercept m. The figure shows isoclines for $m = -6, -4, -2, 0, 2, 4$. It also shows a few solutions. Note the solution that is also an isocline is shown in pink.

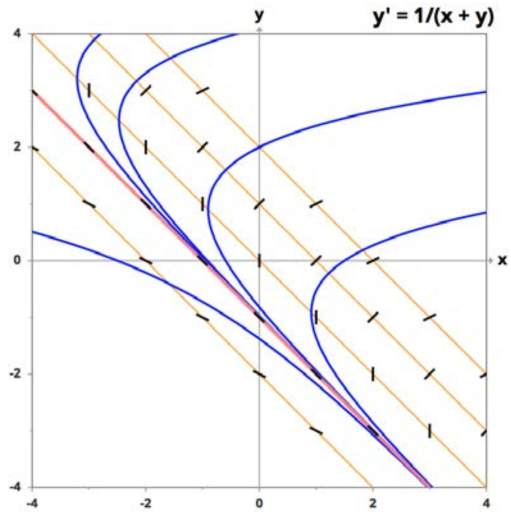


Because these lines all have slope -2, the isocline for slope -2 is also a solution. That is, $y = -2x - 2$ is a solution. It is easy to show this by substituting $y = -2x - 2$ into the DE.

(c) Solution: The isoclines for slope m are the circles $x^2 + y^2 = 1 + m$. These are circles centered at the origin. Since $1 + m$ must be positive there are isoclines for $m \geq -1$.



(d) Solution: The isocline for slope m is $\dfrac{1}{x+y} = m$. For $m \neq 0$ this is equivalent to $x + y = 1/m$. Each isocline is a line slope -1. Several are shown. Several curves that follow the direction field are also shown.

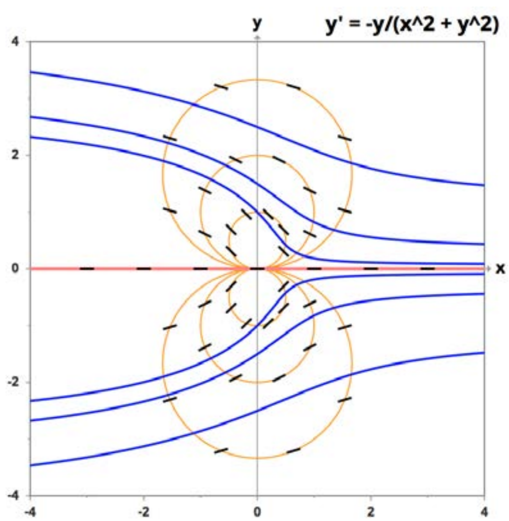


No other integral curve will cross the solution $y = -x - 1$. The existence and uniqueness theorem says that integral curves can't cross in any region where $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous.

10.2. (a) Solution: The isoclines have equation $-y/(x^2 + y^2) = m$.
The nullcline is the line $y = 0$.
If $m \neq 0$, a little algebraic manipulation (completing the square) gives

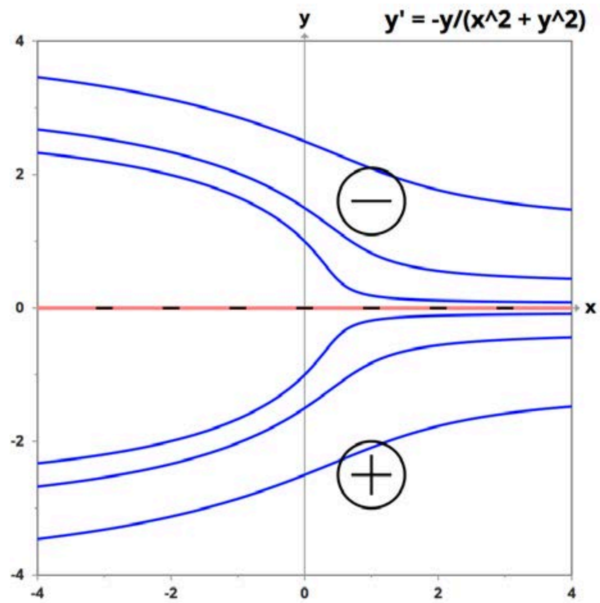
$$\begin{aligned} x^2 + y^2 = -y/m &\Leftrightarrow x^2 + y^2 + y/m = 0 \\ &\Leftrightarrow x^2 + y^2 + y/m + (1/2m)^2 = (1/2m)^2 \\ &\Leftrightarrow x^2 + (y + 1/2m)^2 = (1/2m)^2. \end{aligned}$$

This is clearly the equation of a circle centered on the y-axis and going through the origin.



$y(0) = 1$: In the first quadrant $y' < 0$. Since $y(x) \equiv 0$ is a solution, the existence and uniqueness theorem says that the integral curve cannot cross the x-axis for $x > 0$. (The theorem fails at the origin, but that is not a problem for $x > 0$.) Thus, starting at $(0, 1)$, the integral curve has negative slope and goes into the first quadrant and cannot cross the x-axis, i.e., must stay positive.

(b) Solution: As in Part (a), the nullcline is the x-axis. We can easily see that, $y' < 0$ when $y > 0$ and $y' > 0$ when $y < 0$. Thus integral curves above the x-axis slope down and those below it slope up.



The sketch is done with the computer, so the integral curves are accurate. A hand sketch would show qualitatively that the curves all go asymptotically to the x-axis.

10.3. Solution: The existence and uniqueness theorem requires the equation in standard form.

$$y' + \frac{b(x)}{a(x)}y = \frac{c(x)}{a(x)}.$$

For this equation, it says that if $b(x)/a(x)$ and $c(x)/a(x)$ are continuous near x_0, then there is a unique solution. Typically, this will be when a, b, c are continuous and $a(x_0) \neq 0$.

Topic 11. Numerical methods for first-order ODEs

Solutions

11.1. (a) Solution: We organize the computation in a table.

n	x_n	y_n	$m = f(x_n, y_n)$	mh
0	0	1.0	-1.0	-0.1
1	0.1	0.9	-0.8	-0.08
2	0.2	0.82	-0.62	-0.062
3	0.3	0.758		

So, $y(0.1) \approx 0.9$, $y(0.2) \approx 0.82$, $y(0.3) \approx 0.758$.

Taking the second derivation: $y'' = 1 - y' = 1 - x + y$. Since this is positive in the region near $(0, 1)$, the integral curve is concave up. Therefore, Euler's method gives an underestimate for $y(0.3)$.

(You could see the concavity graphically by drawing isoclines.)

(b) Solution: We only have to take one step. Following the Topic 11 notes:

$k_1 = f(x_0, y_0) = f(0, 1) = -1$. So, $x_a = x_0 + h = 0.1$, $y_a = y_0 + k_1 h = 0.9$,

$k_2 = f(x_a, y_a) = f(0.1, 0.9) = -0.8$.

$m = \dfrac{k_1 + k_2}{2} = -0.9$.

Take step: $x_1 = x_0 + h = 0.1$, $y_1 = y_0 + mh = 0.91$.

In Part (a) we decided that the Euler estimate was too low. This method increased the estimate from 0.9 to 0.91. So RK2 did correct it in the right direction.

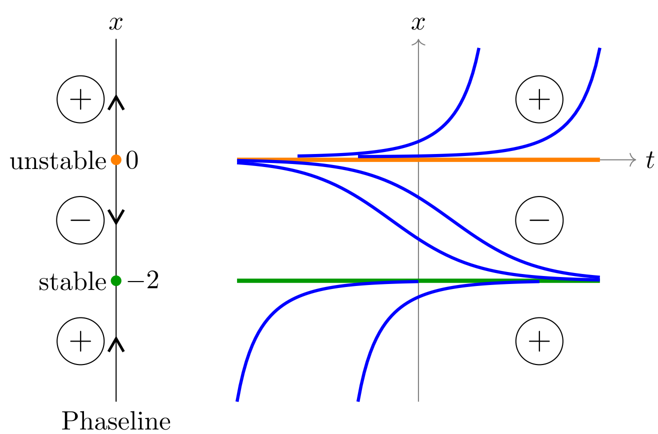
Topic 12. Autonomous DEs and bifurcation diagrams

Solutions

12.1. (a) Solution: 1. Critical points: $x' = x^2 + 2x = 0$, so $x = 0, -2$.

2. Checking signs for x':
when $x > 0$, $x' > 0$;
when $-2 < x < 0$, $x' < 0$;
when $x < -2$; $x' > 0$.

This gives the following phase line and sketch of solutions.

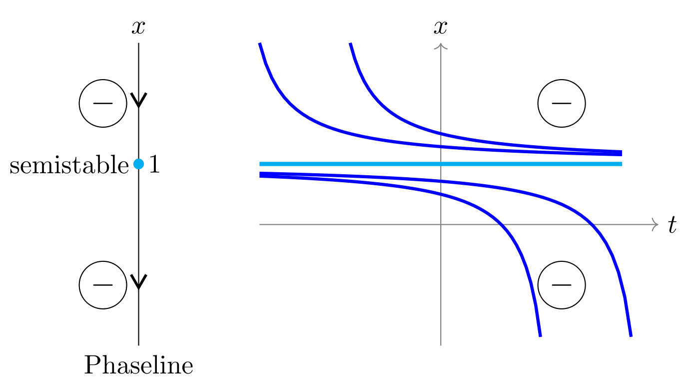


(b) Solution: 1. Critical points: $x' = -(x-1)^2 = 0$, so $x = 1$.

2. Checking signs for x':
when $x > 1$, $x' < 0$;
when $x < 1$; $x' < 0$.

(So $x = 1$ is a semistable critical point.)

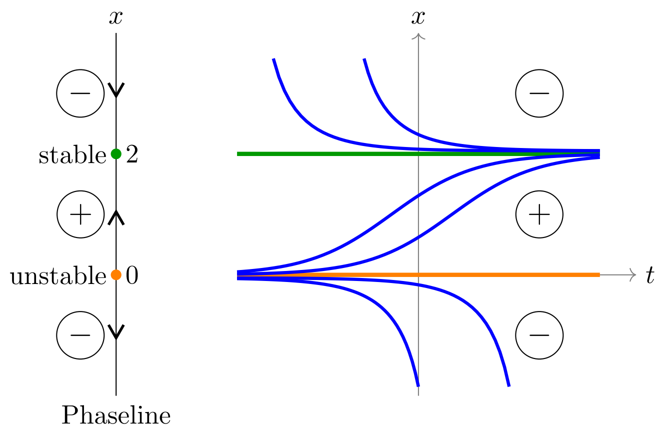
This gives the following phase line and sketch of solutions.



(c) Solution: 1. Critical points: $x' = 2x - x^2 = 0$, so $x = 0, 2$.

2. Checking signs for x':
when $x > 2$, $x' < 0$;
when $0 < x < 2$; $x' > 0$. when $x < 0$, $x' < 0$.

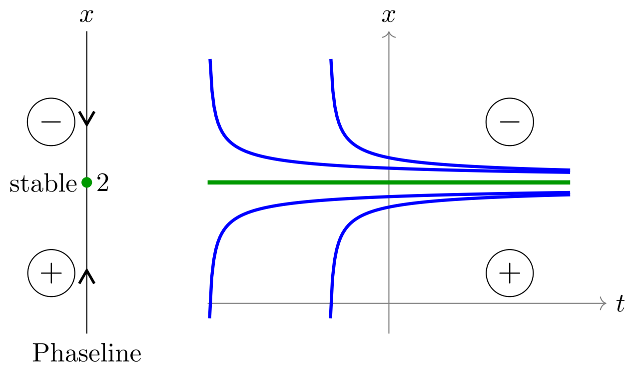
This gives the following phase line and sketch of solutions.



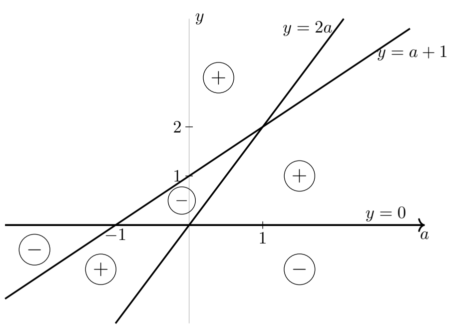
(d) **Solution:** 1. Critical points: $x' = (2-x)^3 = 0$, so $x = 2$.

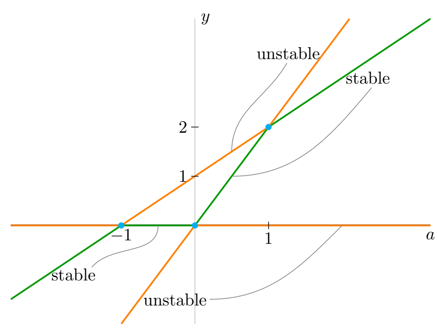
2. Checking signs for x':
when $x > 2$, $x' < 0$;
when $x < 2$, $x' > 0$.

This gives the following phase line and sketch of solutions.



12.2. **(a)** **Solution:** Critical points: $y' = y(y-a-1)(y-2a) = 0 \quad \Rightarrow y = 0, y = a+1, y = 2a$.





The three lines showing the critical points divide the ay-plane into 6 regions. We mark the regions pluses and minuses to indicate where y' is positive or negative. We then translate this into a color coded bifurcation diagram. (We also label the various branches as stable or unstable.)

(b) **Solution:** Sustainable means a positive, stable critical point. So, this system is sustainable for $a > 0$.

(c) **Solution:** Bifurcation points at $a = 0$, $a = 1$, $a = -1$.

Topic 13. Linear algebra: matrices, vector spaces, linearity

Solutions

13.1. (a) Solution: There are many possible choices, e.g., $(1/\sqrt{2}, 1/\sqrt{2})$, $(3/5, 4/5)$, $(1/2, 1/2, 1/2, 1/2)$.

(b) Solution: Again we have many choices.

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ -6 \end{bmatrix} \text{ etc.}$$

(c) Solution: This is important. To pick out a column, you use a column vector consisting of all 0's and one 1.

$$\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(d) Solution: This is similar to Part (c): We need the row vector $\mathbf{w} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

13.2. (a) Solution: You have to check the set is closed under scaling and addition. This set is a vector space. Closure under scaling and addition is just our old friend 'superposition of homogeneous solutions'.

(b) Solution: Not a vector space because not closed under either scaling or addition. For example, the entries of $(1, 0, 0)$ sum to 1, but the entries of $(3, 0, 0)$ do not.

(c) Solution: Is a vector space. It is easy to check that if $f(0) = f(\pi) = 0$ and $g(0) = g(\pi) = 0$ the $x(t) = c_1 f(t) + c_2 g(t)$ satisfies the same properties.

(d) Solution: Is a vector space. Practically by definition, the set of *all* linear combinations is closed under scaling and addition.

13.3. (a) Solution: The system is in balance, i.e., the volume of fluid in each compartment stays constant. The system is

$$\begin{aligned} x' &= -4\frac{x}{V_1} + 3\frac{y}{V_2} + 1 \cdot 3 = -0.4x + 0.6y + 3 \\ y' &= 4\frac{x}{V_1} - 6\frac{y}{V_2} + 2 \cdot 2 \quad = 0.4x - 1.2y + 4 \end{aligned}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -0.4 & 0.6 \\ 0.4 & -1.2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{or } \mathbf{x}' = A\mathbf{x} + \mathbf{K}.$$

(b) Solution: The problem says to try a constant solution $\mathbf{x} = \mathbf{C}$. Substituting in the DE gives

$$\mathbf{C} = -A^{-1}\mathbf{K} = -\frac{1}{0.24} \begin{bmatrix} -1.2 & -0.6 \\ -0.4 & -0.4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \boxed{\frac{1}{0.24} \begin{bmatrix} 6 \\ 2.8 \end{bmatrix}}.$$

Topic 14. Linear algebra: row reduction and subspaces

Solutions

14.1. Solution: (a) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$. All columns are multiples of the first implies rank $= 1$.

(b) $\begin{bmatrix} 2 \end{bmatrix}$. A 1x1 (nonzero) matrix has rank 1.

(c) $\begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix}$. Two independent columns implies rank $= 2$.

14.2. (a) Solution: (i) No: Pivots are not in descending rows, i.e., first pivot is in Row 2 and second is in Row 1.
(ii) Yes: Pivot columns are all 0's except for one 1. Pivots occur in descending rows.
(iii) Yes: same reason.
(iv) No: There is a zero row (Row 2) followed by a pivot row (Row 3).
(v) Yes: Simple, but satisfies the definition.

(b) Solution: (i) $\begin{bmatrix} 1 \end{bmatrix}$
(ii) $\begin{bmatrix} 1 & 1 \end{bmatrix}$
(iii) Subtract Row 1 from Row 2: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$
(iv)

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{\text{Multiply } R_1 \text{ by } -1/2} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{\text{Subtract } R_2 = R_1 \text{ from } R_2} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{\text{Multiply } R_2 \text{ by } -2/3} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{\text{Subtract } R_2 \text{ from } R_3} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & -\frac{4}{3} \end{bmatrix} \xrightarrow{\text{Multiply } R_3 \text{ by } -3/4} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\text{Add } 2/3 \times R_3 \text{ to } R_2} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Add } 1/2 \times R_2 \text{ to } R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

14.3. (a) Solution: We have to solve the equation

$$x_1\,\mathbf{v_1} + x_2\,\mathbf{v_2} + x_3\,\mathbf{v_3} + x_4\,\mathbf{v_4} = \mathbf{0} \Leftrightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

That is, we need to find a non-zero vector in the null space of the matrix. (Note, because there are more columns than rows, we know in advance that there will be at least one free column.) We do this by row reduction:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow[R_1 = R_1 - R_3]{R_2 = R_2 - R_3} \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Setting the free variable $x_4 = 1$ we get the following null vector $\begin{bmatrix} 1 & 1 & -4 & 1 \end{bmatrix}^T$. Thus the equation called for is

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Solution: (i) Row reduction for A:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \xrightarrow{\text{Swap } R_1 \text{ and } R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Stop and observe: The third column is twice the second minus the first. That's true in the original matrix as well! The fourth column is the 3 times the second minus twice the first. That's also true in the original matrix!

Setting one free variable to 1 and the other to 0 we get two basis vectors for the null space:

$$x_3 = 1,\, x_4 = 0 \to \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 = 0,\, x_4 = 1 \to \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix},$$

For B:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow{\text{Swap } R_1 \text{ and } R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[R_4 = R_4 - 3R_1]{R_3 = R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow[R_4 = R_4 + 2R_2]{R_3 = R_3 + 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The first two variables are pivotal and the third is free. Set the free variable equal to 1 and solve for the pivot variables to find the basis vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. (Any nonzero multiple of this vector is another basis for the null space.)

(ii) We need a particular solution to $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The systematic approach is to use the augmented matrix formed by the pivot columns of A and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then solve using row reduction:

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{array}\right] \xrightarrow{\text{Swap } R_1 \text{ and } R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{array}\right]$$
$$\xrightarrow{R_3 = R_3 + R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

The pivot variables are x_1 and x_2. The last augmented matrix above shows the solution is $x_1 = -1$, $x_2 = 1$. Combining this with the free variables $x_3 = 0$, $x_4 = 0$, we have a particular solution $x_p = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Thus the general solution is

$$\mathbf{x} = \mathbf{x_p} + \mathbf{x_h} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Note: in this case, you might have just noticed that the Column 2 minus Column 1 equals $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. This would have given you the solution without further computation.

(iii) We need a particular solution to $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Again, the difference of the first two columns works; so the general solution is

$$\mathbf{x} = \mathbf{x_p} + \mathbf{x_h} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

(Or using the systematic approach would have been a sure fire way to find the solution.)

(c) Solution: (i) This is easy to do directly: all such vectors are multiples of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, so that by itself forms a basis of this subspace.

Since the basis has one element, the dimension of the subspace is 1.

(ii) A matrix representation of this relation is $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{x} = 0$. This is already in reduced echelon form! The first variable is pivotal, the last three are free. We get a basis: $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Since the basis has 3 elements, the subspace has dimension 3.

(iii) Now there are two equations in four unknowns. We get the matrix equation

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$$

We use row reduction to get a basis for the null space:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}.$$

The first two variables are pivotal and the last two are free. A basis is $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Looking back at the original equations, this makes sense, doesn't it?

Since the basis has 2 elements, the subspace has dimension 2.

(d) Solution: (i) We need the rank of the matrix to be 2. If $c \neq 0$ then no matter what d is there will be pivots in all three rows and the rank will be 3: so $c = 0$. The rank will still be 3 unless the third row is a linear combination of the first two. Said differently, row reduction has to bring the last row to 0, so we must have $d = 2$.

(ii) If the null space is two dimensional, then there are 2 free columns. Since there are 4 columns, this means there are 2 pivot columns. Part (i) essentially asked when there were exactly two pivot columns. Thus the answer here is the same as in Part (i)

Topic 15. Linear algebra: transpose, inverse, determinant

Solutions

15.1. **Solution: (a)** $\det \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = 1$; $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$.

(b) $\det \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = 1$ (by cross-hatch, or because the determinant of an upper-triangular matrix is the product of the diagonal entries). We find the inverse using row reduction on an augmented matrix. We don't show all the intermediate steps. The steps are: subtract $c\times\text{row}_3$ from row_2; subtract $b\times\text{row}_3$ from row_1; subtract $a\times\text{row}_2$ from row_1

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1 = R_1 - b\cdot R_3]{R_2 = R_2 - c\cdot R_3} \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 = R_1 - a\cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

So the inverse is $\begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$.

(c) $\det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 2$ by crosshatch. So we expect a 2 in the denominator of the inverse.

We find the inverse using row reduction on an augmented matrix. We don't show all the intermediate steps. The steps are: swap row_1 and row_2; subtract row_1 from row_3; subtract row_2 from row_3; scale row_3 by -1/2; subtract row_3 from row_2; subtract row_3 from row_1

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Swap } R_1 \text{ and } R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \xrightarrow{R_3 = -\frac{1}{2}\cdot R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow[R_1 = R_1 - R_3]{R_2 = R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

so the inverse is $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

The original matrix was symmetric. Is it an accident that the inverse is also symmetric?

(d) $\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 1\cdot 2\cdot 3\cdot 4 = 24$. The inverse is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$.

15.2. (a) Solution:

$$\begin{aligned} R(\alpha)R(\beta) &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\cos\alpha\sin\beta + \sin\alpha\cos\beta) \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \\ &= R(\alpha+\beta). \end{aligned}$$

Geometrically, $R(\theta)$ rotates vectors in the plane by θ radians counterclockwise.

(b) Solution: $\det R(\theta) = (\cos\theta)^2 + (\sin\theta)^2 = 1$. $R(\theta)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R(-\theta)$.

Topic 16. Linear algebra: eigenvalues, diagonalization

Solutions

16.1. (a) Solution: (i) The eigenvalues of upper or lower triangular matrices are the diagonal entries: so for A we get eigenvalues 1 and 2.

Eigenvectors (vectors in $\text{Null}(A-\lambda I)$):

For $\lambda = 1$: $A - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$; So $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector, i.e. in $\text{Null}(A-\lambda I)$. Any multiple is also an eigenvector.

For $\lambda = 2$: $A - \lambda I = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or any multiple is an eigenvector.

For B, the eigenvalues are 3 and 4.

For $\lambda = 3$: $B - \lambda I = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$; So $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a basic eigenvector.

For $\lambda = 4$: $B - \lambda I = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$; So $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basic eigenvector.

(ii) AA: Directly from the definition of eigenvalue/eigenvector we can see that the eigenvalues of AA are the squares of those of A and the eigenvectors are those of A. For example, if $A\mathbf{v} = \lambda\mathbf{v}$, then

$$AA\mathbf{v} = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda \cdot \lambda\mathbf{v} = \lambda^2\mathbf{v}.$$

Thus AA has eigenvalues 1 and 4.

Multiplying: $AB = \begin{bmatrix} 4 & 4 \\ 2 & 8 \end{bmatrix}$

Characteristic equation: $\begin{vmatrix} 4-\lambda & 4 \\ 2 & 8-\lambda \end{vmatrix} = \lambda^2 - 12\lambda + 24 = 0$

Eigenvalues (roots): $6 \pm \sqrt{12}$.

(iii) If $A\mathbf{x} = \lambda\mathbf{x}$, then $(cA)\mathbf{x} = cA\mathbf{x} = c\lambda\mathbf{x}$, so $c\lambda$ is an eigenvalue of cA. We see that the eigenvalues of cA are exactly c times the eigenvalues of A.

(iv) $A + B = \begin{bmatrix} 4 & 1 \\ 1 & 6 \end{bmatrix}$ has characteristic equation $\begin{vmatrix} 4-\lambda & 1 \\ 1 & 6-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 23 = 0$. So the eigenvalues of $A + B$ are $5 \pm \sqrt{2}$.

We see that the eigenvalues of $A + B$ are not just the sum of the eigenvalues of A and B.

(b) Solution: The characteristic polynomial of matrix A is $\det(A - \lambda I)$.

Remember: The determinant of a diagonal or triangular matrix is just the product of the diagonal elements.

(i) $\begin{vmatrix} 1-\lambda & a \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2$.

(ii) $\begin{vmatrix} 1-\lambda & a & b \\ 0 & 1-\lambda & c \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3$.

(iii) $\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (-\lambda)^3 + 1 + 1 - (-\lambda) - (-\lambda) - (-\lambda) = -\lambda^3 + 3\lambda + 2.$

(iv) $\begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(3-\lambda)(4-\lambda).$

16.2. (a) Solution: (i) A is upper triangular so its eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$.

Next, we find the corresponding basic eigenvectors (basis of Null$(A - \lambda I)$):

$\lambda_1 = 1$: $A - \lambda_1 I = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$, so we can take $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\lambda_2 = 3$: $A - \lambda_1 I = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$, so we can take $\mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus, $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $\Rightarrow A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

For B: Characteristic equation $\det \begin{bmatrix} 1-\lambda & 1 \\ 3 & 3-\lambda \end{bmatrix} = \lambda^2 - 4\lambda = 0$.

So the eigenvalues are $\lambda = 0$, 4.

Basic eigenvectors (basis of Null$(B - \lambda I)$):

$\lambda = 0$: $A - \lambda I = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\lambda = 4$: $A - \lambda I = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Thus, $S = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$, $S^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ $\Rightarrow B = S\Lambda S^{-1}$

(ii) These follow directly from Part (i).

$A^3 = S\Lambda^3 S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 27 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

$A^{-1} = S\Lambda^{-1} S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

(b) Solution: A has rank 1 implies its null space has dimension 9. Since the null space consists of all eigenvectors with eigenvalue 0, $\lambda = 0$ is an eigenvalue repeated 9 times.

This leaves one more eigenvalue to find. Since the

$$\text{tr}A = 5 = \text{sum of the eigenvalues}$$

the last eigenvalue is $\lambda = 5$.

(c) Solution: Characteristic equation: $\det(A - \lambda I) = \lambda^2 + 4\lambda + 4 = 0 \Rightarrow \lambda = -3, -1$.

Basic eigenvectors (basis of Null$(A - \lambda I)$):

$\lambda = -3$: $\quad A - \lambda I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow$ take $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\lambda = -1$: $\quad A - \lambda I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow$ take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Since we want the columns of S to be unit vectors we normalize the eigenvectors

$\lambda_1 = -3$: $\mathbf{v_1} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \qquad \lambda_2 = -1$: $\mathbf{v_2} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

So, $S = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $\Lambda = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$, $S^{-1} = S^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Finally $A = S\Lambda S^{-1}$.

16.3. (a) Solution: (i) $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

(ii) We could do this by inspection but let's do it computationally.

Characteristic equation: $\lambda^2 + 2\lambda = 0 \Rightarrow \lambda = 0, -2$.

Basic eigenvectors (basis of $\text{Null}(A - \lambda I)$):

$\lambda_1 = 0$: $A - \lambda_1 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. We can take $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda_2 = -2$: $A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. We can take $\mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(iii) Using the eigenvalues and eigenvectors, the solution to the system of DEs is

$$\mathbf{u}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The initial conditions conditions are

$$\mathbf{u}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix} \qquad \Rightarrow c_1 = 20,\ c_2 = 10.$$

We conclude $\begin{bmatrix} v(1) \\ w(1) \end{bmatrix} = \begin{bmatrix} 20 + 10e^{-2} \\ 20 - 10e^{-2} \end{bmatrix}$

(iv) Looking at the solution to the system, we see that as t goes to infinity both v and w go to 20.

(b) Solution: To give it a name, call the coefficient matrix $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$<

We start by finding the general solution to this system using the method of eigenvalues and eigenvectors.

Characteristic equation: $\det(B - \lambda I) = \lambda^2 + 1 = 0$. So, $\lambda = \pm i$.

Basic eigenvectors (basis of $\text{Null}(B - \lambda I)$):

$\lambda = i$: $\quad B - \lambda i = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$. We can take $\mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

Complex modal solution: $\mathbf{z}(t) = e^{it}\begin{bmatrix}1\\ i\end{bmatrix} = \begin{bmatrix}\cos(t) + i\sin(t)\\ -\sin(t) + i\cos(t)\end{bmatrix}$

Two real-valued solutions (real and imaginary parts of $\mathbf{z}$):

$$\mathbf{x_1}(t) = \begin{bmatrix}\cos(t)\\ -\sin(t)\end{bmatrix} \qquad \mathbf{x_2}(t) = \begin{bmatrix}\sin(t)\\ \cos(t)\end{bmatrix}.$$

General solution:

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t) = \begin{bmatrix}c_1\cos(t) + c_2\sin(t)\\ -c_1\sin(t) + c_2\cos(t)\end{bmatrix}.$$

$\mathbf{x}(t)$ is on the unit circle if $|\mathbf{x}(t)| = 1$. So we compute

$$|\mathbf{x}(t)| = (c_1\cos(t) + c_2\sin(t))^2 + (-c_1\sin(t) + c_2\cos(t))^2 = c_1^2 + c_2^2$$

This shows that the solutions that trace out the unit circle are those with $c_1^2 + c_2^2 = 1$. Note that the simplest of these is $\mathbf{x_1}(t)$, which traces the unit circle in a clockwise direction starting at $(1, 0)$.

Topic 17. Matrix methods of solving systems of DEs

Solutions

17.1. (a) Solution: If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then $\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]\begin{bmatrix}\mathbf{v}\\ \mathbf{0}\end{bmatrix} = \begin{bmatrix}A\mathbf{v}\\ \mathbf{0}\end{bmatrix} = \begin{bmatrix}\lambda\mathbf{v}\\ \mathbf{0}\end{bmatrix}$. This shows that $\begin{bmatrix}\mathbf{v}\\ \mathbf{0}\end{bmatrix}$ is an eigenvector of C with eigenvalue λ.

Likewise, if $\mathbf{w}$ is an eigenvector of B with eigenvalue μ, then $\begin{bmatrix}\mathbf{0}\\ \mathbf{w}\end{bmatrix}$ is an eigenvector of C with eigenvalue μ.

Conclusion: The eigenvalues of C are $\lambda_1, \dots, \lambda_m, \mu_1 \dots, \mu_n$.

(b) Solution: We take a general eigenvector and see what properties it must have. Suppose $\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix}$ is an eigenvector of C with eigenvector λ. Then, $\left[\begin{array}{c|c} 0 & I \\ \hline A & 0 \end{array}\right]\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix} = \begin{bmatrix}\mathbf{w}\\ A\mathbf{v}\end{bmatrix} = \lambda\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix}$.

This shows that $\mathbf{w} = \lambda\mathbf{v}$ and $A\mathbf{v} = \lambda\mathbf{w}$. Substituting the first expression for $\mathbf{w}$ in the second equation gives

$$A\mathbf{v} = \lambda\mathbf{w} = \lambda^2\mathbf{v}.$$

This shows that $\mathbf{v}$ is an eigenvector of A with eigenvector λ^2.

Conclusion: the eigenvalues of C are $\lambda = \pm\sqrt{\mu}$, where μ is an eigenvalue of A. If $\mathbf{v}$ is eigenvector of A with eigenvalue μ, then $\begin{bmatrix}\mathbf{v}\\ \sqrt{\mu}\mathbf{v}\end{bmatrix}$ and $\begin{bmatrix}\mathbf{v}\\ -\sqrt{\mu}\mathbf{v}\end{bmatrix}$ are eigenvectors of C with eigenvalues $\sqrt{\mu}$ and $-\sqrt{\mu}$ are respectively.

17.2. (a) Solution: The sum of the x coefficients is 5, as is the sum of the y coefficeints. This shows that the total number of rabbits grows according to $(x+y)' = 5(x+y)$.

(b) Solution: The rate that rabbits jump from Jones' to McGregor is given by the coefficient 2 of $2x$ in the equation for y'. Likewise, the rate in the opposite direction is given by the coefficient 1 of y. So the rate from Jones' to McGregor is twice that in the other direction.

(c) Solution: The matrix of eigenvectors is $S = \begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}$. So, $S^{-1} = \frac{1}{3}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}$. The diagonal matrix of eigenvalues is $\Lambda = \begin{bmatrix}5 & 0\\ 0 & 2\end{bmatrix}$.

Diagonalization: $A = S\Lambda S^{-1} = \frac{1}{3}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}$.

Decoupling: Call the decoupled coordinates $\mathbf{u} = \begin{bmatrix}u\\ v\end{bmatrix}$. The decoupled coordinates are

$$\mathbf{u} = S^{-1}\mathbf{x} \Leftrightarrow \begin{bmatrix}u\\ v\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{rcl} u &=& x/3 + y/3 \\ v &=& 2x/3 - y/3. \end{array}$$

The decoupled equations are $\mathbf{u}' = \Lambda\mathbf{u} \quad \Leftrightarrow u' = 5u;\ v' = 2v.$

17.3. (a) Solution: First we have to find eigenvalues and eigenvectors.

Characteristic equation: $|A - \lambda I| = \lambda^2 - 1 = 0$. So, $\lambda = 1, -1$.

Eigenvectors (basis of Null$(A - \lambda I)$):

Note: for a 2×2 matrix we don't need to use row reduction to find a basis of the null space of $A - \lambda I$.

$\lambda = 1$: $(A - \lambda I) = \begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda = -1$: $(A - \lambda I) = \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

The modal solutions (also called normal modes) are

$$\mathbf{x_1}(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x_2}(t) = e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The general solution to the system of DEs is $\mathbf{x}(t) = c_1 \mathbf{x_1}(t) + c_2 \mathbf{x_2}(t)$.

(b) Solution: $B - \lambda I = \begin{bmatrix} 4 - \lambda & -3 \\ 8 & -6 - \lambda \end{bmatrix}$

Characteristic equation: $|B - \lambda I| = \lambda^2 + 2\lambda = 0$. So, $\lambda = 0, -2$.

Eigenvectors (basis of Null$(B - \lambda I)$):

$\lambda = 0$: $(B - \lambda I) = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

$\lambda = -2$: $(B - \lambda I) = \begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The modal solutions (also called normal modes) are

$$\mathbf{x_1}(t) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x_2}(t) = e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The general solution to the system of DEs is $\mathbf{x}(t) = c_1 \mathbf{x_1}(t) + c_2 \mathbf{x_2}(t)$.

(c) Solution: $C - \lambda I = \begin{bmatrix} 1 - \lambda & -1 & 0 \\ 1 & 2 - \lambda & 1 \\ -2 & 1 & -1 - \lambda \end{bmatrix}$

We computed the determinant using Laplace expansion along the top row:

Characteristic equation: $|C - \lambda I| = (\lambda - 1)(\lambda - 2)(\lambda + 1) = 0$. So, $\lambda = 1, 2, -1$.

Basic eigenvectors (basis of Null$(C - \lambda I)$):

$\lambda = 1$: $(C - \lambda I) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -2 & 1 & -2 \end{bmatrix}$. We put this into RREF $R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

This has 1 free variable. So the eigenspace (null space of $A - \lambda I$) has one basis vector. We take $\mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$\lambda = 2$: $(C - \lambda I) = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & -3 \end{bmatrix}$. We put this into RREF $R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

This has 1 free variable. So the eigenspace has one basis vector. We take $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

$\lambda = -1$: $(C - \lambda I) = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$. We put this into RREF $R = \begin{bmatrix} 1 & 0 & 1/7 \\ 0 & 1 & 2/7 \\ 0 & 0 & 0 \end{bmatrix}$.

This has 1 free variable. So the eigenspace has one basis vector. We take $\mathbf{v} = \begin{bmatrix} -1 \\ -2 \\ 7 \end{bmatrix}$.

The modal solutions (also called normal modes) are

$$\mathbf{x_1}(t) = e^t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x_2}(t) = e^{2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x_3}(t) = e^{-t} \begin{bmatrix} -1 \\ -2 \\ 7 \end{bmatrix}.$$

The general solution to the system of DEs is $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t) + c_3\mathbf{x_3}(t)$.

17.4. (Complex eigenvalues) **Solution:** Call the coefficient matrix A. $A - \lambda I = \begin{bmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{bmatrix}$.

Characteristic equation: $|A - \lambda I| = \lambda^2 + 4 = 0$. So, $\lambda = \pm 2i$.

Basic eigenvectors (basis of Null$(A - \lambda I)$):

Note: for a 2×2 matrix we don't need to use row reduction to find a basis of the null space of $A - \lambda I$.

$\lambda = 2i$: $A - \lambda I = \begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 5 \\ 1-2i \end{bmatrix}$.

For $\lambda = -2i$, we could just take the complex conjugate of the above vector. BUT, we don't need the second eigenvector.

One complex solution:

$$\mathbf{z}(t) = e^{2it} \begin{bmatrix} 5 \\ 1-2i \end{bmatrix} = (\cos(2t) + i\sin(2t)) \begin{bmatrix} 5 \\ 1-2i \end{bmatrix} = \begin{bmatrix} 5\cos(2t) + i5\sin(2t) \\ \cos(2t) + 2\sin(2t) + i(\sin(2t) - 2\cos(2t)) \end{bmatrix}$$

This gives two real-valued solutions

$$\mathbf{x_1}(t) = \mathrm{Re}(\mathbf{z}) = \begin{bmatrix} 5\cos(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix}$$
$$\mathbf{x_2}(t) = \mathrm{Im}(\mathbf{z}) = \begin{bmatrix} 5\sin(2t) \\ \sin(2t) - 2\cos(2t) \end{bmatrix}$$

The general real-valued solution is $\mathbf{x}(\mathbf{t}) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$.

17.5. (Complex eigenvalues) **Solution:** Call the coefficient matrix A. $A - \lambda I = \begin{bmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{bmatrix}$.

Characteristic equation: $|A - \lambda I| = \lambda^2 - 6\lambda + 25 = 0$. So, $\lambda = 3 \pm 4i$.

Basic eigenvectors (basis of Null$(A - \lambda I)$):

$\lambda = 3 + 4i$: $A - \lambda I = \begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

For $\lambda = 3 - 4i$ we would just take the complex conjugate of the above vector. BUT, we don't need to bother finding this.

One complex solution:

$$\mathbf{z}(t) = e^{(3+4i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{3t}(\cos(4t) + i\sin(4t)) \begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{3t} \begin{bmatrix} \cos(4t) + i\sin(4t) \\ \sin(4t) - i\cos(4t) \end{bmatrix}$$

This gives two real-valued solutions

$$\mathbf{x_1}(t) = \mathrm{Re}(\mathbf{z}) = e^{3t} \begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix}$$
$$\mathbf{x_2}(t) = \mathrm{Im}(\mathbf{z}) = e^{3t} \begin{bmatrix} \sin(4t) \\ -\cos(4t) \end{bmatrix}$$

The general real-valued solution is $\mathbf{x}(\mathbf{t}) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$.

Topic 18. Matrix exponential, exponential and sinusoidal input

Solutions

18.1. Solution: We know the answer is $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$.

In order to compute e^{At}, we need the eigenvalues and eigenvectors of A. We don't show the computation, since you should be able to do it by now. Also, you can easily check our answers. –Why?

The eigenvalues are 5, 1 and these have corresponding eigenvectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Using these we have: $S = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ (matrix of eigenvectors), $\Lambda = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ (diagonal matrix of eigenvalues).

So, $A = S\Lambda S^{-1}$ and $e^{At} = Se^{\Lambda t}S^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$.

The solution to the IVP is

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

18.2. (a) Solution: Instead of using the Exponential Response Formula, we'll essentially rederive it:

We try a solution of the form $\mathbf{x} = e^{2t}\mathbf{C}$. Substitution into the DE gives

$$2e^{2t}\mathbf{C} = e^{2t}A\mathbf{C} + e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Leftrightarrow (2I - A)\mathbf{C} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Leftrightarrow \mathbf{C} = (2I - A)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Computing:

$$\mathbf{C} = \begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

So a particular solution is $\mathbf{x_p}(t) = e^{2t}\mathbf{C} = -\frac{e^{2t}}{2}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

(b) Solution: We write the input as a sum and use superposition. The DE is

$$\mathbf{x}' = A\mathbf{x} + e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We solved the equation $\mathbf{x_1}' = A\mathbf{x_1} + e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in Part (a), where we found a particular solution $\mathbf{x_{1,p}} = -\frac{e^{2t}}{2}\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

So we solve for the second piece, i.e., $\mathbf{x_2}' = A\mathbf{x_2} + e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Using the ERF (it's easy to remember now that we rederived it in Part (a)) we have

$$\mathbf{C} = (3I - A)^{-1}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}4 & -1\\5 & 0\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1\\4\end{bmatrix}.$$

So a particular solution to this part is $\mathbf{x_{2,p}}(t) = \dfrac{e^{3t}}{5}\begin{bmatrix}1\\4\end{bmatrix}$.

By superposition, a particular solution to the original equation is

$$\mathbf{x_p}(t) = \mathbf{x_{1,p}}(t) + \mathbf{x_{2,p}}(t) = -\frac{e^{2t}}{2}\begin{bmatrix}1\\5\end{bmatrix} + \frac{e^{3t}}{5}\begin{bmatrix}1\\4\end{bmatrix}.$$

(c) Solution: Complexify:

$$\mathbf{z}' = \begin{bmatrix}-1 & 1\\-5 & 3\end{bmatrix}\mathbf{z} + e^{it}\begin{bmatrix}1\\0\end{bmatrix}, \text{ where } \mathbf{x} = \text{Im}(\mathbf{z}).$$

The exponential response formula gives a solution of the form $\mathbf{z} = e^{it}\mathbf{C}$, where

$$\mathbf{C} = (iI - A)^{-1}\mathbf{K} = \begin{bmatrix}i+1 & -1\\5 & i-3\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix}$$
$$= \frac{1}{1-2i}\begin{bmatrix}i-3 & 1\\-5 & i+1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{1-2i}\begin{bmatrix}i-3\\-5\end{bmatrix}.$$

We could use amplitude-phase form here, but it is probably easier to use rectangular form to find the imaginary part.

$$\mathbf{z_p}(\mathbf{t}) = \frac{e^{it}}{1-2i}\begin{bmatrix}i-3\\-5\end{bmatrix} = \frac{(\cos(t)+i\sin(t))(1+2i)}{5}\begin{bmatrix}i-3\\-5\end{bmatrix} = \frac{1}{5}\begin{bmatrix}-5\cos(t)+5\sin(t)+i(-5\cos(t)-5\sin(t))\\-5\cos(t)+10\sin(t)+i(-10\cos(t)-5\sin(t))\end{bmatrix}$$

So, $\boxed{\mathbf{x_p}(\mathbf{t}) = \text{Im}(\mathbf{z_p}(\mathbf{t})) = \begin{bmatrix}-\cos(t)-\sin(t)\\-2\cos(t)-\sin(t)\end{bmatrix}}$.

18.3. Solution: We use the exponential response formula to find a particular solution. Rather than memorize the theorem, we will simply rederive it.

To solve $\mathbf{x}' = A\mathbf{x} + e^{at}\mathbf{K}$, we try a solution of the form $\mathbf{x} = e^{at}\mathbf{C}$. Substituting this in the DE and solving for the unknown $\mathbf{C}$ we get

$$ae^{at}\mathbf{C} = e^{at}A\mathbf{C} + e^{at}\mathbf{K} \quad \Rightarrow (aI - A)\mathbf{C} = \mathbf{K} \quad \Rightarrow \mathbf{C} = (aI - A)^{-1}\mathbf{K}.$$

For the given problem we have a superposition of exponential inputs

$$\mathbf{x}' = Ax + e^{-2t}\begin{bmatrix}1\\0\end{bmatrix} + e^{t}\begin{bmatrix}0\\-2\end{bmatrix}.$$

So we apply the theorem to each input separately.

$$\mathbf{C_1} = (-2I - A)^{-1}\begin{bmatrix}1\\0\end{bmatrix} = -\frac{1}{4}\begin{bmatrix}0 & 1\\4 & -3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix} \quad \Rightarrow \mathbf{x_{1,p}}(t) = e^{-2t}\begin{bmatrix}0\\-1\end{bmatrix}.$$
$$\mathbf{C_2} = (I - A)^{-1}\begin{bmatrix}0\\-2\end{bmatrix} = -\frac{1}{4}\begin{bmatrix}3 & 1\\4 & 0\end{bmatrix}\begin{bmatrix}0\\-2\end{bmatrix} = \begin{bmatrix}1/2\\0\end{bmatrix} \quad \Rightarrow \mathbf{x_{2,p}}(t) = \frac{1}{2}e^{t}\begin{bmatrix}1\\0\end{bmatrix}.$$

Now superposition gives a particular solution to the DE:

$$\mathbf{x}(t) = \mathbf{x_{1,p}} + \mathbf{x_{2,p}} = e^{-2t}\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \frac{1}{2}e^{t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Topic 19. Fundamental matrix, variation of parameters

Solutions

19.1. Solution: Let $y = x'$. The equation becomes $y' + p(t)y + q(t)x = f(t)$. We write the pair of equations:

$$\begin{array}{lllll} x' & = & & y & \\ y' & = -q(t)x & - & p(t)y & +f(t) \end{array} \quad \Leftrightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

19.2.

(a) Solution: A set of vectors are independent if no one of them is a linear combination of the others (equivalently, if no nontrivial linear combination of all of them is 0). For two vectors this means the neither is a *constant* mutiple of the other. This is clear for the two vector functions given.

(b) Solution: The Wronskian is defined as the determinant of the matrix with these vectors as columns:

$$W(\mathbf{x_1}, \mathbf{x_2}) = \begin{vmatrix} \mathbf{x_1} & \mathbf{x_2} \end{vmatrix} = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2$$

(c) Solution: Following the hint we let $\Phi(t) = \begin{bmatrix} \mathbf{x_1} & \mathbf{x_2} \end{bmatrix} = \begin{bmatrix} t & t^2 \\ 1 & 2t \end{bmatrix}$.

Since $\Phi' = A\Phi$, we know $A(t) = \Phi'(t)\Phi^{-1}(t)$. This is just a calculation. We use the answer in Part (b) to help compute the inverse.

$$A = \Phi'\Phi^{-1} = \begin{bmatrix} 1 & 2t \\ 0 & 2 \end{bmatrix} \left(\frac{1}{t^2} \begin{bmatrix} 2t & -t^2 \\ -1 & t \end{bmatrix} \right) = \frac{1}{t^2} \begin{bmatrix} 0 & t^2 \\ -2 & 2t \end{bmatrix}$$

Thus the system is $\mathbf{x}' = \dfrac{1}{t^2} \begin{bmatrix} 0 & t^2 \\ -2 & 2t \end{bmatrix} \mathbf{x}$.

(d) Solution: The existence and uniqueness theorem requires that $A(t)$ is continuous. Since $A(t)$ (given in Part (c)) is not continuous at $t = 0$, the conclusion of the theorem may not hold when $t = 0$. That is, two different solutions might take the same value at $t = 0$.

19.3. Solution: First we solve the homogeneous equation $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x}$.

Eigenvalues: (char. eq.) $\begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda = 2, -3$.

For eigenvectors we want a basis of the null space of $A - \lambda I$.

$\lambda = 2$: $(A - \lambda I) = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}$. By inspection we can take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as an eigenvector.

$\lambda = -3$: $(A - \lambda I) = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}$. By inspection we can take $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ as an eigenvector.

Thus we have two solutions: $\mathbf{x_1}(\mathbf{t}) = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x_2} = e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

Next we solve the given inhomogeneous equation using variation of parameters.

Fundamental matrix $\Phi = \begin{bmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{bmatrix} \Rightarrow |\Phi| = -5e^{-t} \Rightarrow \Phi^{-1} = -\frac{e^t}{5}\begin{bmatrix} -4e^{-3t} & -e^{-3t} \\ -e^{2t} & e^{2t} \end{bmatrix}$.

Variation of parameters formula:

$$\mathbf{x} = \Phi(t)\left(\int \Phi(t)^{-1}\mathbf{F}(\mathbf{t})\,dt + \mathbf{c}\right) = \Phi(t)\left(\int -\frac{e^t}{5}\begin{bmatrix} -4e^{-3t} & e^{-3t} \\ -e^{2t} & e^{2t} \end{bmatrix}\begin{bmatrix} e^{-2t} \\ -2e^t \end{bmatrix}dt + \mathbf{c}\right).$$

Side work to compute the integral:

$$\int -\frac{e^t}{5}\begin{bmatrix} -4e^{-3t} & -e^{-3t} \\ -e^{2t} & e^{2t} \end{bmatrix}\begin{bmatrix} e^{-2t} \\ -2e^t \end{bmatrix}dt = \frac{1}{5}\int \begin{bmatrix} 4e^{-4t} - 2e^{-t} \\ e^t + 2e^{4t} \end{bmatrix}dt = \frac{1}{10}\begin{bmatrix} -2e^{-4t} + 4e^{-t} \\ 2e^t + e^{4t} \end{bmatrix}.$$

Thus, $\mathbf{x}(t) = \Phi(t)\cdot\left(\frac{1}{10}\begin{bmatrix} -2e^{-4t} + 4e^{-t} \\ 2e^t + e^{4t} \end{bmatrix} + \mathbf{c}\right) = \boxed{\begin{bmatrix} e^t/2 \\ -e^{-2t} \end{bmatrix} + c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -4 \end{bmatrix}}$.

19.4. Solution: First we solve the homogeneous equation $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\mathbf{x}$.

Eigenvalues: (char. eq.) $\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = 1, -1$.

For eigenvectors we want a basis of the null space of $A - \lambda I$.

$\lambda = 1$: $(A - \lambda I) = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$. By inspection we can take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as an eigenvector.

$\lambda = -1$: $(A - \lambda I) = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$. By inspection we can take $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ as an eigenvector.

Thus we have two solutions: $\mathbf{x_1} = e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x_2} = e^{-t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Next we solve the given inhomogeneous equation.

Fundamental matrix $\Phi = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix} \Rightarrow |\Phi(t)| = 2$ and $\Phi^{-1} = \frac{1}{2}\begin{bmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{bmatrix}$.

Variation of parameters formula:

$$\mathbf{x} = \Phi(t)\left(\int \Phi(t)^{-1}\mathbf{F}(\mathbf{t})\,dt + \mathbf{c}\right) = \Phi(t)\left(\int \frac{1}{2}\begin{bmatrix} 3e^{-t} & e^{-t} \\ -e^t & e^t \end{bmatrix}\begin{bmatrix} e^t \\ -e^t \end{bmatrix}dt + \mathbf{c}\right).$$

Side work to compute the integral:

$$\int \frac{1}{2}\begin{bmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{bmatrix}\begin{bmatrix} e^t \\ -e^t \end{bmatrix}dt = \int \begin{bmatrix} 2 \\ -e^{2t} \end{bmatrix}dt = \begin{bmatrix} 2t \\ e^{2t}/2 \end{bmatrix}.$$

Thus,

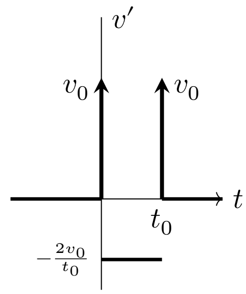
$$\mathbf{x}(t) = \Phi(t)\cdot\left(\begin{bmatrix} 2t \\ e^{2t}/2 \end{bmatrix} + \mathbf{c}\right) = \boxed{\begin{bmatrix} 2te^t - e^t/2 \\ 2te^t - 3e^t/2 \end{bmatrix} + c_1 e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}}.$$

Topic 20. Step and delta functions.

Solutions

20.1. (Integration) **Solution:** The interval of integration contains 0 and 5, but not -1 or 20. So only the $\delta(t)$ and $\delta(t-5)$ terms contribute to the integral. Their contributions are 3 and 25 (t^2 evaluated at 5). So the value of the integral is 28.

20.2. (Differentiation) **Solution:** $v'(t) = v_o\delta(t) + v_0\delta(t-t_0) + \begin{cases} 0 & \text{for } t < 0 \\ -\frac{2v_0}{t_0} & \text{for } 0 < t < t_0 \\ 0 & \text{for } t > t_0 \end{cases}$



20.3. **Solution: (a)** The pre-initial conditions are $x(0^-) = 0$, $x'(0^-) = 0$.

For $t < 0$: The input $\delta(t) = 0$. So the DE with initial conditions is

$$2x'' + 2x' = 0; \qquad x(0^-) = 0,\ x'(0^-) = 0.$$

It is easy to see the solution to this is $x(t) = 0$.

For $t > 0$: The post-initial conditions are $x(0^+) = 0$, $x'(0^+) = 1/2$.

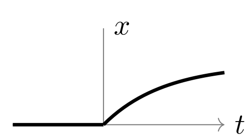
On this interval, the input $\delta(t) = 0$. So the DE with initial conditions is

$$2x'' + 2x' = 0; \qquad x(0^+) = 0,\ x'(0^+) = 1/2.$$

The general homogeneous solution is $x(t) = c_1 + c_2e^{-t}$. Using the post-IC to find c_1 and c_2, we get $x(t) = 1/2 - e^{-t}/2$.

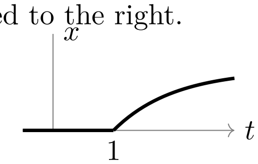
Putting the pieces together we have

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{2} - \frac{1}{2}e^{-t} & \text{for } t > 0. \end{cases}$$



(b) This is exactly the same as Part (a) except time is shifted to the right.

$$x(t) = \begin{cases} 0 & \text{for } t < 1 \\ \frac{1}{2} - \frac{1}{2}e^{-(t-1)} & \text{for } t > 1. \end{cases}$$



Topic 21. Fourier series: basics.

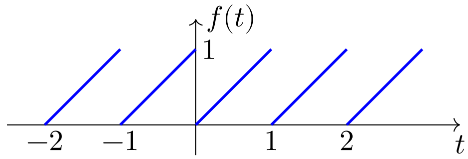
Solutions

21.1. Solution: (a) $\sin(\pi t/3)$ has period 6.

(b) $|\sin(t)|$ has period π.

(c) $\cos(3t)$ has period $2\pi/3$ so $\cos^2(3t)$ has period $\boxed{\pi/3}$.

21.2. Solution: Here is the sketch:



We have the half-period $L = 1/2$. The integrals for the Fourier coefficient should be over one full period. In this case, it seems easier to integrate from 0 to 1 rather than from $-1/2$ to $1/2$.

$$a_0 = \frac{1}{1/2}\int_0^1 t\,dt = 2\left[\frac{t^2}{2}\right]_0^1 = 1$$

$$a_n = \frac{1}{1/2}\int_0^1 t\cos(2n\pi t)\,dt = 2\left[\frac{t\sin(2n\pi t)}{2n\pi} + \frac{cos(2n\pi t)}{(2n\pi)^2}\right]_0^1 \quad \text{(by parts or by table lookup)}$$
$$= 0$$

$$b_n = \frac{1}{1/2}\int_0^1 t\sin(2n\pi t)\,dt = 2\left[\frac{-t\cos(2n\pi t)}{2n\pi} + \frac{\sin(2n\pi t)}{(2n\pi)^2}\right]_0^1 \quad \text{(by parts or by table lookup)}$$
$$= -\frac{1}{n\pi}$$

So, $f(t) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n\cos(2n\pi t) + \sum_{n=1}^{\infty} b_n\sin(2n\pi t) = \boxed{\frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n\pi t)}{n}}$.

Topic 22. Fourier series introduction continued.

Solutions

22.1. Solution: This is an even function, so we only need to compute the cosine coefficients (a_n). We don't show all the details of the integrations.

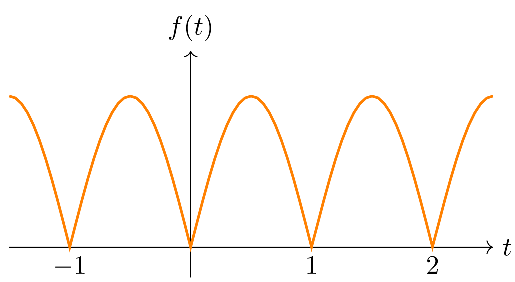
We have the half-period $L = 1/2$. In this case, it is easiest to integrate over a full period $[0, 1]$ rather than use the doubling trick for even functions.

$$a_0 = \frac{1}{1/2}\int_0^1 \sin(\pi t)\,dt = -2\left[\frac{\cos(\pi t)}{\pi}\right]_0^1 = \frac{4}{\pi}$$

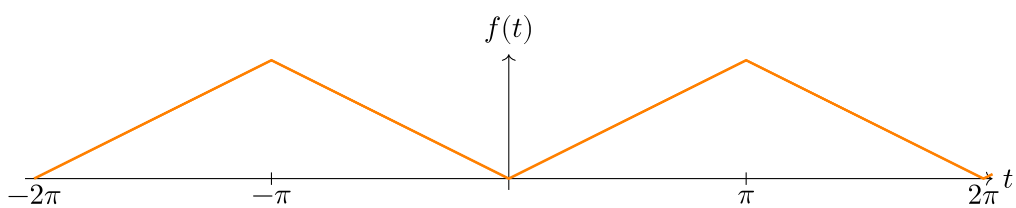
$$a_n = \frac{1}{1/2}\int_0^1 \sin(\pi t)\cos(2n\pi t)\,dt = 2\int_0^1 \frac{\sin(\pi t + 2n\pi t) + \sin(\pi t - 2n\pi t)}{2}\,dt = -\frac{4}{\pi(4n^2-1)}$$

This integral can be done using the formula $\sin(\pi t)\cos(2n\pi t) = \dfrac{\sin(\pi t + 2n\pi t) + \sin(\pi t - 2n\pi t)}{2}$ or by table lookup.

So, $f(t) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\cos(2n\pi t)}{4n^2-1}$



22.2. Solution: We start by graphing $f(t)$.



This is an even function with half-period $L = \pi$. So, $b_n = 0$ and

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t\,dt = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} t\cos(nt)\,dt = \begin{cases} -4/n^2\pi & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$$

(The integral can be done by parts or by table lookup.)

So, $f(t) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\cdot\dfrac{\cos(t)}{1} - \dfrac{4}{\pi}\cdot\dfrac{\cos(3t)}{3^2} - \dfrac{4}{\pi}\cdot\dfrac{\cos(5t)}{5^2}\,\ldots = \dfrac{\pi}{2} - \dfrac{4}{\pi}\displaystyle\sum_{n \text{ odd}}\frac{\cos(nt)}{n^2}$

Topic 23. Sine and cosine series; calculation tricks.

Solutions

23.1.

(a) Solution: We have $L = 1$. To compute the sine coefficients, the integral for b_n is computed using integration by parts (or table lookup).

$$b_n = 2\int_0^1 (1-x)\sin(n\pi x)\,dx = 2\left[-\frac{\cos(n\pi x)}{n\pi} + \frac{x\cos(n\pi x)}{n\pi} - \frac{\sin(n\pi x)}{n^2\pi^2}\right]_0^1 = \frac{2}{n\pi}$$

So, over the interval $0 < x < 1$,

$$f(x) = \frac{2}{\pi}\sin(\pi x) + \frac{2}{2\pi}\sin(2\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \frac{2}{4\pi}\sin(4\pi x) + ... = \frac{2}{\pi}\sum_1^\infty \frac{\sin(n\pi x)}{n}.$$

(b) Solution: As in Part (a) we have $L = 1$. We compute the cosine coefficients. The integral for a_n is computed using integration by parts.

$$a_0 = 2\int_0^1 1 - x\,dx = 1$$

$$a_n = 2\int_0^1 (1-x)\cos n\pi x\,dx = 2\left[\frac{\sin(n\pi x)}{n\pi} - \frac{x\sin(n\pi x)}{n\pi} - \frac{\cos(n\pi x)}{n^2\pi^2}\right]_0^1 = \begin{cases} 4/(n^2\pi^2) \text{ for } n \text{ odd} \\ 0 \text{ for } n \text{ even}\end{cases}$$

So, over the interval $0 < x < 1$,

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\cos(\pi x) + \frac{4}{3^2\pi^2}\cos(3\pi x) + \frac{2}{5^2\pi^2}\cos(5\pi x) + ... = \frac{1}{2} + \frac{4}{\pi^2}\sum_{\text{odd}} \frac{\cos(n\pi x)}{n^2}.$$

Topic 24. Linear ODEs with periodic input.

Solutions

24.1. (a) Solution: For an undamped system $mx'' + kx = f(t)$, the natural frequency is $\omega_0 = \sqrt{k/m}$. So the natural frequency of the system $2x'' + 10x = f(t)$ is $\omega_0 = \sqrt{5}$.

The function $f(t)$ is odd with period 2, so $f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t)$. Since none of the frequencies $n\pi$ equals $\sqrt{5}$, there is no pure resonance.

(b) Solution: The natural frequency of the system $x'' + 4\pi^2 x = f(t)$ is $\omega_0 = 2\pi$.

The function $f(t)$ is odd with period 2, so $f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t)$. This means that the $n = 2$ term will cause resonance as long as $b_2 \neq 0$. So we need to compute the value of b_2.

We have $L = 1$ and $f(t)$ is odd, so

$$b_2 = 2\int_0^1 2t\sin(2\pi t)\,dt = \left[-\frac{4t\cos(2\pi t)}{2\pi} + \frac{4\sin(2\pi t)}{(2\pi)^2}\right]_0^1 = \frac{-4}{2\pi}.$$

Since $b_2 \neq 0$, there is pure resonance.

(c) Solution: The natural frequency of the system $x'' + 9x = f(t)$ is $\omega_0 = 3$.

The function $f(t)$ is odd with period 2π, so $f(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$. So the $n = 3$ term will cause resonance as long as $b_3 \neq 0$.

We know that $f(t)$ is the period 2π, odd square wave, so

$$f(t) = \frac{4}{\pi}\sum_{n\,\text{odd}} \frac{\sin(nt)}{n}.$$

Since $b_3 = \frac{4}{3\pi} \neq 0$, there is pure resonance.

24.2. Solution: First we compute the Fourier series for $f(t)$. Since $f(t)$ is odd with period 2π, we have

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(nt),$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} 2t\sin(nt)\,dt = \frac{2}{\pi}\left[-\frac{2t\cos(nt)}{n} + \frac{2\sin(nt)}{n^2}\right]_0^{\pi} = (-1)^{n+1}\frac{4}{n}$$

So the DE is: $\quad x'' + x' + 3x = 4\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\sin(nt)}{n}.$

We'll solve for each piece first and then use superposition. Let

$$x_n'' + x_n' + 3x_n = (-1)^{n+1}\frac{4}{n}\sin(nt).$$

We can solve this using the sinusoidal response formula (SRF). First, compute $P(in)$ in polar form. $(P(r) = r^2 + r + 3)$

$$P(in) = 3 - n^2 + in; \quad |P(in)| = \sqrt{(3-n^2)^2 + n^2}; \quad \boxed{\phi(n) = \text{Arg}(P(in)) = \tan^{-1} n/(3-n^2) \text{ in Q1 or Q2}}.$$

The SRF gives: $\boxed{x_{n,p}(t) = (-1)^{n+1}\frac{4\sin(nt - \phi(n))}{n|P(in)|} = (-1)^{n+1}\frac{4\sin(nt-\phi(n))}{n\sqrt{(3-n^2)^2+n^2}}}.$

Superposition:

$$\boxed{x_p(t) = \sum_{n=0}^{\infty} x_{n,p} = 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(nt-\phi(n))}{n\sqrt{(3-n^2)^2+n^2}}}.$$

24.3. (a) Solution: $f(t)$ is the same function as in problem 2. Its Fourier series is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(nt), \text{ where } b_n = (-1)^{n+1}\frac{4}{n}$$

Let's consider $f(t)$ to be the input. This does not change the arithmetic, but it will allow us to use the language of gain and resonance.

The system has natural frequency $\omega_0 = 3$. Since it is lightly damped, the resonant frequency will be close to ω_0. So we expect the $\sin(3t)$ term to have the biggest gain. This tells us to expect that the output term with the biggest amplitude will be either the $\sin(3t)$ term or a nearby one.

It is easy enough to compute the output amplitude for each term:

$$\text{Input term: } (-1)^{n+1}\frac{4}{n}\sin(nt) \Rightarrow \text{ output: } (-1)^{n+1}\frac{4\sin(nt-\phi(n))}{n|P(in)|} = (-1)^{n+1}\frac{4\sin(nt-\phi(n))}{n\sqrt{(18-2n^2)^2+(0.1n)^2}}.$$

Looking at the first few output amplitudes we get

$$\begin{aligned} n = 1: &\text{ amplitude} = \frac{4}{\sqrt{16^2+0.01}} \approx 0.25 \\ n = 2: &\text{ amplitude} = \frac{4}{2\sqrt{10^2+0.04}} \approx 0.20 \\ n = 3: &\text{ amplitude} = \frac{4}{3\sqrt{.09}} \approx 4.44 \\ n = 4: &\text{ amplitude} = \frac{4}{4\sqrt{14^2+.16}} \approx 0.07. \end{aligned}$$

This shows that the $n = 3$ term has the biggest output amplitude.

(b) Solution: We need to compute the Fourier series for $f(t)$. Since it is odd with period 2 we have

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t),$$

where

$$b_n = 2\int_0^1 (t-t^2)\sin(n\pi t)\,dt = \begin{cases} 8/(n\pi)^3 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

Let's consider $f(t)$ to be the input. This does not change the arithmetic, but it will allow us to use the language of gain and resonance.

The system has natural frequency $\omega_0 = \sqrt{10}$. Since it is lightly damped, the resonant frequency will be close to ω_0. So we expect the $\sin(\pi t)$, i.e., $n = 1$, term to have the biggest gain. This tells us to expect that the output term with the biggest amplitude will be either the $\sin(\pi t)$ term or a nearby one.

It is easy enough to compute the output amplitude for each term:

$$\text{Input term: } \frac{8}{(\pi n)^3}\sin(n\pi t) \Rightarrow \text{ output: } \frac{8\sin(n\pi t - \phi(n))}{(\pi n)^3 |P(in\pi)|} = \frac{8\sin(n\pi t - \phi(n))}{(\pi n)^3 \sqrt{(30 - 3\pi^2 n^2)^2 + \pi^2 n^2}}.$$

Since both the gain and the input amplitude are biggest for the $n = 1$ term, we know that the $n = 1$ term has the biggest output amplitude.

Topic 25. PDEs; separation of variables.

Solutions

25.1. Solution: First, let's identify all the parts of the problem:

PDE and range. $u_t = u_{xx}$ for $0 \le x \le 10$; $t > 0$.

Boundary conditions (BC). $u_x(0, t) = u_x(10, t) = 0$. (Notice the derivatives.)

Initial conditions (IC). $u(x, 0) = 4x$.

Here are the steps to the solution:

Step 1. Find separated solutions to the PDE.

Step 2. Find the modal solutions, i.e., the sepatated solutions that also satisfy the BC.

Step 3. The general solution is a linear combination of the modal solutions.

Step 4. Use the initiial conditions to determine the values of the coefficients in the general solution.

Step 1. (Find separated solutions to the PDE.)

Trial solution to PDE: $u(x, t) = X(x)T(t)$ (i.e., guess a separated solution).

Plug into the PDE: $X(x)T'(t) = X''(x)T(t) \quad \Rightarrow \dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)}$.

Observation: The left side is a function of x and the right is a function of t. Since x and t are independent variables, both sides must be constant. Call the constant $-\lambda$. Therefore,

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -\lambda.$$

Rearranging these equations we get

$$X'' + \lambda X = 0, \qquad T' + \lambda T = 0.$$

These equations are easy to solve. We have three cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, $T(t) = ce^{-\lambda t}$. So,

$$u(x, t) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x) \cdot ce^{-\lambda t}.$$

Case (ii) $\lambda < 0$: $X(x) = ae^{\sqrt{|\lambda|}x} + be^{-\sqrt{|\lambda|}x}$, $T(t) = ce^{-\lambda t}$. So,

$$u(x, t) = X(x)T(t) = (ae^{\sqrt{|\lambda|}x} + be^{-\sqrt{|\lambda|}x}) \cdot ce^{-\lambda t}.$$

Case (iii) $\lambda = 0$: $X(x) = a + bx$, $T(t) = c$. So,

$$u(x, t) = (a + bx) \cdot c.$$

Step 2. (Find the modal solutions, i.e., the separated solutions that also satisfy the boundary conditions.)

For the separated solution $u(x, t) = X(x)T(t)$, our boundary conditions become

$$u_x(0, t) = X'(0)T(t) = 0 \quad \text{and} \quad u_x(10, t) = X'(10)T(t) = 0.$$

Since we want nontrivial solutions, we must have $X'(0) = 0$ and $X'(10) = 0$.

We look at each case in turn.

Case (i): $X'(0) = \sqrt{\lambda} b = 0, \quad X'(10) = -\sqrt{\lambda} a \sin(10\sqrt{\lambda}) + \sqrt{\lambda} b \cos(10\sqrt{\lambda}) = 0.$

These imply $b = 0$ and either $a = 0$ or $\sin(10\sqrt{\lambda}) = 0$.

The case $b = 0$, $a = 0$ gives a trivial solution, so we ignore it. Therefore, we must have $\sin(10\sqrt{\lambda}) = 0$, i.e., $10\sqrt{\lambda} = n\pi$ for some positive integer n. We conclude: for each positive integer n,

$$u_n(x,t) = a_n \cos\left(\frac{n\pi\, x}{10}\right) \cdot e^{-\left(\frac{n\pi}{10}\right)^2 t}$$

satisfies both the PDE and BC.

Case (ii): Easy algebra shows that, in this case, only trivial separated solutions satisfy the BC.

Case (iii): $X'(0) = X'(10) = b = 0 \quad \Rightarrow \quad X(x) = a.$

Conclusion: The constant solution

$$u_0(x,t) = \frac{a_0}{2}$$

satisfies both PDE and BC. (We write it as $a_0/2$ for when it is the constant term in a Fourier series.)

Step 3. Use superposition to get the general solution.

The general solution to the PDF and BC is:

$$u(x,t) \;=\; u_0 + \sum_{n=1}^{\infty} u_n(x,t) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{10}\right) \cdot e^{-\left(\frac{n\pi}{10}\right)^2 t}$$

Step 4. (Use the IC to compute the values of the coefficients)

The IC give

$$u(x,0) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{10}\right) = 4x \qquad \text{on } [0,10].$$

That is, we have a Fourier cosine series for $f(x) = 4x$. You can compute this cosine series directly, or notice that the even period 20 extension is a triangle wave. We found that

$$4x \;=\; 20 - \sum_{n \text{ odd}} \frac{160}{(n\pi)^2} \cos\left(\frac{n\pi}{10}\, x\right).$$

This gives us the values of a_n and the solution to the PDE with BC and IC:

$$u(x,t) \;=\; 20 - \sum_{n \text{ odd}} \frac{160}{(n\pi)^2} \cos\left(\frac{n\pi x}{10}\right) e^{-\left(\frac{n\pi}{10}\right)^2 t} \qquad \text{for } x \text{ in } [0,10].$$

25.2. Solution: We follow the same steps as in Problem 1.

PDE and range. $y_{tt} = 4y_{xx}$ for $0 \le x \le \pi;\ \ t > 0$

Boundary conditions (BC). $y(0,t) = y(\pi,t) = 0$

Initial conditions (IC). $y(x,0) = \frac{1}{10}\sin(2x)$; $y_t(x,0) = 0$.

Step 1. (Find separated solutions to the PDE.)
Trial solution to PDE: $y(x,t) = X(x)T(t)$ (i.e., guess a separated solution).

Plug into PDE: $X(x)T(t)'' = 4X(x)''T(t)$.

Algebra: $\dfrac{X(x)''}{X(x)} = \dfrac{T(t)''}{4T(t)} = -\lambda \quad \Rightarrow X'' + \lambda X = 0,\ T'' + 4\lambda T = 0.$

(As in Problem 1, we have a function of x equals a function of t. Since x and t are independent, both functions must be constant.)

We have three cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, $T(t) = c\cos(2\sqrt{\lambda}t) + d\sin(2\sqrt{\lambda}t)$. So,

$$y(x,t) = X(x)T(t) = (a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)) \cdot (c\cos(2\sqrt{\lambda}t) + d\sin(2\sqrt{\lambda}t)).$$

Case (ii) $\lambda < 0$: Ignore, since it only has trivial solutions to the PDE and BC.

Case (iii) $\lambda = 0$: $X(x) = a + bx$, $T(t) = c + dt$. So,

$$y(x,t) = X(x)T(t) = (a + bx) \cdot (c + dt)$$

Step 2. (Find the modal solutions, i.e., the separated solutions that also satisfy the BC.)

For the separated solution $y(x,t) = X(x)T(t)$, our boundary conditions become

$$y(0,t) = X(0)T(t) = 0 \quad \text{and} \quad y(\pi,t) = X(\pi)T(t) = 0.$$

Since we want nontrivial solutions we must have $X(0) = 0$ and $X(\pi) = 0$.

Looking at the 3 cases:

Case (i): $X(0) = a = 0$, $X(\pi) = a\cos(\sqrt{\lambda}\pi) + b\sin(\sqrt{\lambda}\pi)$.

The nontrivial solutions have $a = 0$ and $\sqrt{\lambda} = n$ for some positive integer n.

Conclusion: For each positive integer n,

$$y_n(x,t) = \sin(nx) \cdot (c_n\cos(2nt) + d_n\sin(2nt))$$

satisfies both the PDE and BC.

Case (ii): Ignore.

Case (iii): $X(0) = a = 0$ $X(\pi) = a + b\pi = 0$. Thus, $a = 0$, $b = 0$, i.e., there are only trivial solutions in this case.

So, for Step 2, only case (i) gives nontrivial modal solutions.

Step 3. (Use superposition to get the general solution.)

By superpositioning all the modal solutions from Step 2 we have

$$y(x,t) = \sum_{n=1} y_n(x,t) = \sum_{n=1} \sin(nx) \cdot (c_n\cos(2nt) + d_n\sin(2nt))$$

is the general solution to the PDE and BC.

Step 4. Use the IC to determine the values of the coefficients.

$$y(x,0) = \sum c_n \sin(nx) = \frac{1}{10}\sin(2x) \text{ on } [0,\pi]$$

That is, the coefficients c_n are the Fourier sine coefficients of the function $\frac{1}{10}\sin(2x)$. Since this is already written as a sine series, we have

$$c_n = \begin{cases} 1/10 & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

Using the other IC:

$$y_t(x,0) = \sum \sin(nx)\cdot d_n 2n = 0 \ \Rightarrow\ d_n = 0 \text{ for all } n.$$

We have solved the PDE with BC and IC:

$$y(x,t) = \sum c_n \sin(nx)\cos(2nt) = \boxed{\frac{1}{10}\sin(2x)\cos(4t).}$$

25.3. Solution: We follow the same steps as in Problems 1 and 2, but with much less commentary.

Step 1. (Find separated solutions to the PDE)

Guess $y(x,t) = X(x)T(t)$ (separated solution).

Plug into PDE: $X(x)T(t)'' = 100X(x)''T(t) \quad \Rightarrow \dfrac{X(x)''}{X(x)} = \dfrac{T(t)''}{100T(t)} = -\lambda$ constant.

Thus, $X'' + \lambda X = 0$, $T'' + \lambda 100T = 0$.

Case (i) $\lambda > 0$: $X(x) = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$, $\quad T(t) = c\cos(10\sqrt{\lambda}t) + d\sin(10\sqrt{\lambda}t)$.

Case (ii) Ignore. Never produces nontrivial modal solutions.

Case (iii) $\lambda = 0 \Rightarrow X(x) = a + bx$, $T(t) = c + dt$.

Step 2. (Find the modal solutions.)

Boundary conditions for nontrivial solutions: $X(0) = 0$ and $X(1) = 0$.

Case (i): $X(0) = a = 0$, $X(1) = a\cos(\sqrt{\lambda}) + b\sin(\sqrt{\lambda}) = 0$.

Solving: Nontrivial solutions when $a = 0$, $\sqrt{\lambda} = n\pi$ for some positive integer n.

Conclusion: For each positive integer n

$$y_n(x,t) = \sin(n\pi x)\cdot(c_n\cos(10n\pi t) + d_n\sin(10n\pi t))$$

satisfies both the PDE and BC, i.e., is a modal solution.

Case (ii): Ignore.

Case (iii): $X(0) = a = 0$, $X(1) = a + b = 0$.

Solving: $a = 0$, $b = 0$, i.e., only trivial solutions in this case.

Step 3. (Use superposition to get the general solution.)

The general solution to the PDE and BC is

$$y(x,t) = \sum_{n=1} y_n(x,t) = \sum_{n=1} \sin(n\pi x) \cdot (c_n \cos(10n\pi t) + d_n \sin(10n\pi t)).$$

Step 4. Use the IC to determine the values of the coefficients.

$$y(x,0) = \sum c_n \sin(n\pi x) = 0 \text{ on } [0,1]$$

So the coefficients $c_n = 0$ for all n.

Using the other IC:

$$y_t(x,0) = \sum \sin(n\pi x) \cdot d_n 10n\pi = x.$$

That is, we have a Fourier sine series for $f(x) = x$ on [0, 1] with the coefficients written in a slightly messy way.

You can compute that the sine series for $f(x) = x$ on [0,1] is

$$f(x) = x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x).$$

Thus,

$$\sum \sin(n\pi x) \cdot d_n 10n\pi = x = \sum_{1}^{\infty} \frac{2(-1)^{n+1}}{\pi n} \quad \Rightarrow d_n = \frac{(-1)^{n+1}}{5\pi^2 n^2}$$

Putting it together,

$$y(x,t) = \sum_{1}^{\infty} \frac{(-1)^{n+1}}{5\pi^2 n^2} \sin(10n\pi t) \sin(n\pi x).$$

Topic 27. Qualitative behavior of linear systems.

Solutions

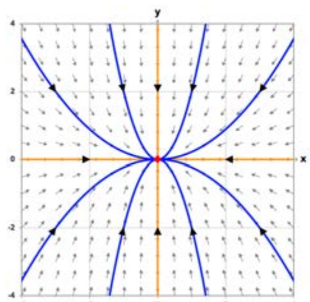
27.1. (a) Solution: This is easy to solve: we find

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} x = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{or} \quad x(t) = c_1 e^{-t},\ y = c_2 e^{-2t}.$$

The modal solutions $c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $c_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ have trajectories that are straight rays heading towards the origin.

Because both eigenvalues are negative, every trajectory goes asymptotically to the origin. The trajectories approach the origin asymptotically tangent to the mode with eigenvalue -1. In the other direction (as $t \to -\infty$) they become asymptotically parallel to the mode with eigenvalue -2.

Here is a phase portrait. The direction is indicated by the arrows showing the vector field.



The critical point is a nodal sink. It is dynamically asymptotically stable.

(b) Solution: The only change would be the direction of increasing time. All the arrows should be reversed.

The critical point is a nodal source. It is dynamically unstable.

27.2. (a) Solution: Coefficient matrix: $A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$.

Characteristic equation: $\begin{vmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = 0$. So, $\lambda = \pm 1$.

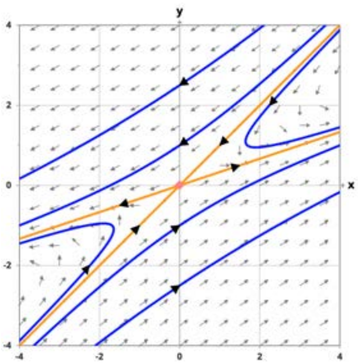
One positive and one negative eigenvalue implies the critical point is a saddle. It's easy to find basic eigenvectors (basis of Null$(A - \lambda I)$):

$\lambda = 1$: $\quad A - \lambda I = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}$, so take $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

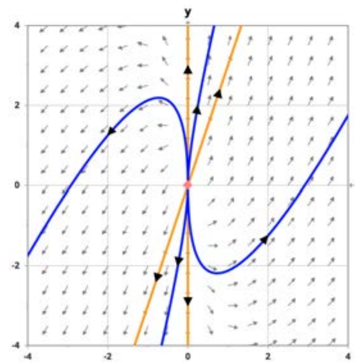
$\lambda = -1$: $\quad A - \lambda I = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}$, so take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The phase portrait is below.

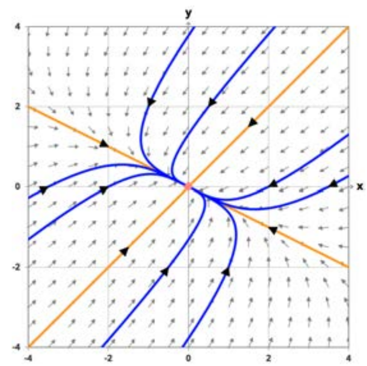
The equilibrium at the origin is dynamically unstable.



Problem 2a: saddle



Problem 2b: nodal source



Problem 2c: nodal sink

(b) Solution: Coefficient matrix: $A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$.

The eigenvalues are the diagonal entries: $\lambda = 2,\ 1$.

Positive distinct eigenvalues imply the critical point is a nodal source. It's easy to compute eigenvectors:

$\lambda = 2$: $A - \lambda I = \begin{bmatrix} 0 & 0 \\ 3 & -1 \end{bmatrix}$, so take $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$\lambda = 1$: $A - \lambda I = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$, so take $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The phase portrait is above.

The equilibrium at the origin is dynamically unstable.

(c) Solution: Coefficient matrix: $A = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}$.

Characteristic equation: $\lambda^2 + 5\lambda + 4 = 0$. So, $\lambda = -1,\ -4$.

Negative distinct eigenvalues imply the critical point is a nodal sink. It's easy to compute eigenvectors:

$\lambda = -1$: $A - \lambda I = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}$, so take $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$\lambda = -4$: $A - \lambda I = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$, so take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The phase portrait is above.

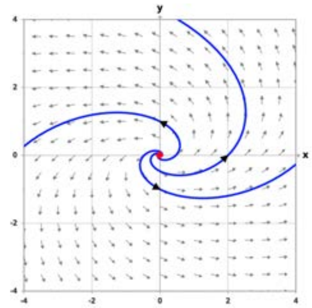
The equilibrium at the origin is dynamically stable.

(d) Solution: Coefficient matrix: $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$.

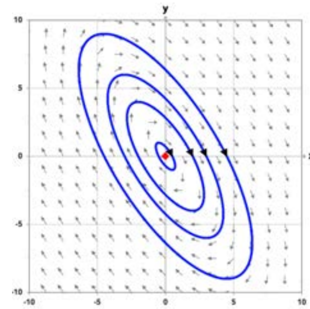
Characteristic equation: $\lambda^2 - 2\lambda + 3 = 0$. So, $\lambda = 1 \pm \sqrt{2}\, i$.

Complex eigenvalues with positive real parts imply the critical point is a spiral source. The positive entry in the lower left of the matrix implies the spirals turn counterclockwise. The phase portrait is below.

The equilibrium at the origin is dynamically unstable.



Problem 2d: spiral source (CCW)



Problem 2e: center (CW)

(e) Solution: Coefficient matrix: $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$.

Characteristic equation: $\lambda^2 + 1 = 0$. So, $\lambda = \pm i$.

Pure imaginary eigenvalues imply the critical point is a center. The -2 entry in the lower left of the matrix implies the loops turn clockwise. The phase portrait is above.

The equilibrium at the origin is an edge case in terms of dynamic stability.

27.3. (a) Solution: To find the companion system, we let $y = x'$. The companion system is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(b) Solution: The companion matrix is $\begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}$. This has characteristic equation $\lambda^2 + k/m = 0$. This has roots $\lambda = \pm\sqrt{k/m}\, i$. Pure imaginary eigenvalues indicate the critical point at the origin is a center.

We call this an edge case or dynamically marginally stable.

(c) Solution: The eigenvalues are $\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$. Since b is small relative to m and k, the eigenvalues are complex with negative real parts. This means the citical point at the origin is a spiral sink. This is a dynamically stable equilibrium.

Looking at the coefficient matrix, the $-k/m$ in the lower left tells us the spirals turn in a clockwise direction.

(d) Solution: The eigenvalues are $\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2}$. Since b is large relative to m and k, the eigenvalues are real and negative. This means the citical point at the origin is a nodal sink. This is a dynamically stable equilibrium.

(e) Solution: No! The determinant of the matrix is k/m, which is positive. Saddles have a negative determinant. Said differently, we know the eigenvalues are either real and negative or complex. This means the critical point can't have one positive and one negative eigenvalue.

Topic 28. Qualitative behavior of nonlinear systems.

Solutions

28.1. **(a)** **Solution:** Critical points are where $x' = 0$ and $y' = 0$.

$x^2 - y^2 = 0 \Rightarrow x = \pm y$.

$x - xy = x(1-y) = 0 \Rightarrow x = 0$ or $y = 1$.

Thus the critical points are $(0,0)$, $(1,1)$, $(-1,1)$.

(b) **Solution:** Critical points are where $x' = 0$ and $y' = 0$.

$1 - x + y = 0 \Rightarrow y = x - 1$. Substituting for y in the second equation gives $y + 2x^2 = x - 1 + 2x^2 = 0$. This factors giving $x = 1/2$ or $x = -1$.

Thus the critical points are $(1/2, -1/2)$, $(-1,-2)$.

28.2. **(a)** **Solution:** The companion system has $y = x'$. So,

$$\begin{aligned} x' &= y \\ y' &= -x - a(x^2-1)y. \end{aligned}$$

For the critical points, the first equation shows $y = 0$. Using this in the second shows $-x = 0$. Thus the only critical point is $(0,0)$.

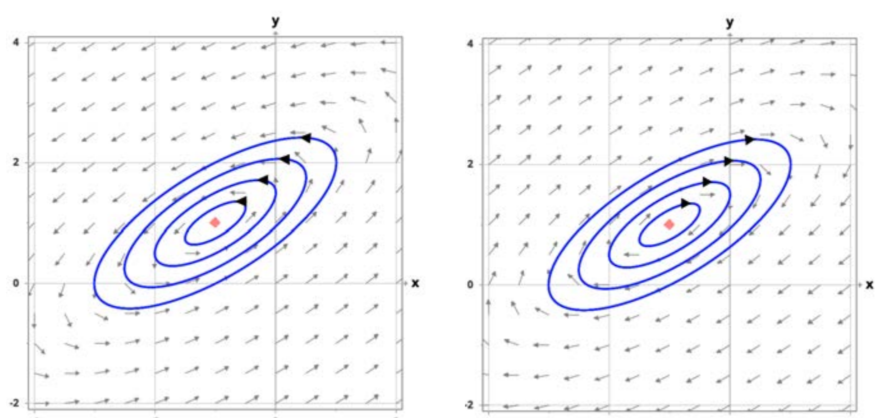
(b) **Solution:** The companion system has $y = x'$. So,

$$\begin{aligned} x' &= y \\ y' &= x^2 + y - 1. \end{aligned}$$

For the critical points, the first equation shows $y = 0$. Using this in the second shows $x^2 = 1$ or $x = \pm 1$. Thus the critical points are $(1,0)$, $(-1,0)$.

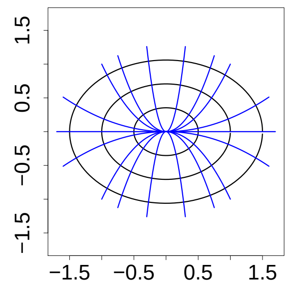
28.3. **(a)** **Solution:** The vector field associated with the original autonomous system is $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$. The new system's vector field is just the negative of this. Thus all the trajectories are the same, except the arrows are reversed.

The figures below illustrate this. The system on the right simply reverses the direction of the vector field and trajectories.



Left: $x' = f(x,y)$, $y' = g(x,y)$, right: $x' = -f(x,y)$, $y' = -g(x,y)$

(b) Solution: The vector field $\begin{bmatrix} g \\ -f \end{bmatrix}$ is orthogonal to $\begin{bmatrix} f \\ g \end{bmatrix}$. Thus the trajectories of the new system are everywhere orthogonal to those of the original system.



Orthogonal trajectories

28.4. (a) Solution: This is just a way of saying that autonomous systems are time invariant. We will go through the proof of this carefully. It is mostly an exercise in taking care with the names of things.

First, by assumption $x'(t) = f(x(t), y(t))$ and $y'(t) = g(x(t), y(t))$.

This is true for any value of the argument t. In particular, if we replace t by $t - t_0$ this becomes

$$x'(t - t_0) = f(x(t - t_0), y(t - t_0)) \text{ and } y'(t - t_0) = g(x(t - t_0), y(t - t_0)). \tag{1}$$

With the help of Equation 1 we get

$$\begin{aligned} \tilde{x}'(t) &= \frac{d}{dt} x(t - t_0) = x'(t - t_0) = f(x(t - t_0), y(t - t_0)) = f(\tilde{x}(t), \tilde{y}(t)) \\ \tilde{y}'(t) &= \frac{d}{dt} y(t - t_0) = y'(t - t_0) = g(x(t - t_0), y(t - t_0)) = g(\tilde{x}(t), \tilde{y}(t)). \end{aligned}$$

We've shown that $(\tilde{x}, \tilde{y})$ is a solution. QED

Shifting by t_0 does not change the trajectory. It merely changes the time the trajectory passes through each of its points. That is, the relationship between the trajectories is that they are the same with different initial points.

(b) Solution: First, suppose we have two trajectories that intersect at a point (a, b). The problem asks us to show that the trajectories are the same.

Call the two solutions $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$. By assumption, there are values t_1 and t_2 such that $(x_1(t_1), y_1(t_1)) = (x_2(t_2), y_2(t_2)) = (a, b)$.

Using Part (a), we have two time shifted solutions

$$(\tilde{x}_1(t), \tilde{y}_1(t)) = (x_1(t + t_1), y_1(t + t_1)) \text{ and } (\tilde{x}_2(t), \tilde{y}_2(t)) = (x_2(t + t_2), y_2(t + t_2)).$$

At $t = 0$ we have

$$\begin{aligned} (\tilde{x}_1(0), \tilde{y}_1(0)) &= (x_1(t_1), y_1(t_1)) = (a, b) \\ (\tilde{x}_2(0), \tilde{y}_2(0)) &= (x_2(t_2), y_2(t_2)) = (a, b) \end{aligned}$$

This shows that $(\tilde{x}_1, \tilde{y}_1)$ and $(\tilde{x}_2, \tilde{y}_2)$ are both solutions satisfying the same initial condition. Therefore, by the existence and uniqueness theorem they must be the same solution.

Since the solutions are the same, so are their trajectories. But these are also the trajectories of the original (unshifted) solutions. That is, if the trajectories of (x_1, y_1) and (x_2, y_2) have a single point in common, then they are the same trajectory. This is what we were supposed to show.

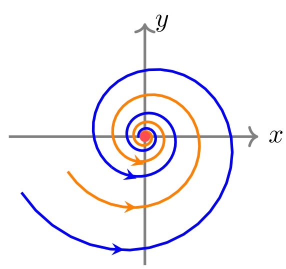
28.5. Solution: Let $f(x,y) = x - y + xy$ and $g(x,y) = 3x - 2y - xy$. Clearly $f(0,0) = 0$ and $g(0,0) = 0$, so $(0,0)$ is a critical point. The linearization at $(0,0)$ is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The characteristic equation is $\lambda^2 + \lambda + 1 = 0$. This has roots $\dfrac{-1 \pm \sqrt{3}\, i}{2}$.

Complex roots with negative real part mean the critical point is a linearized spiral sink. A spiral sink is dynamically stable.

In the figure, we are zoomed in around the origin. The phase portrait does not look like spirals away from the origin.



Note: Since the linearized spiral sink is *structurally stable*, the nonlinear system also looks like a spiral sink near the critical point. This means our sketch is qualitatively correct near the critical point.

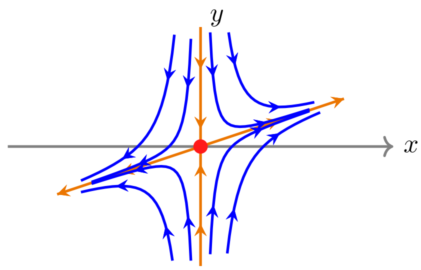
28.6. Solution: Let $f(x,y) = x + 2x^2 - y^2$ and $g(x,y) = x - 2y - x^3$. Clearly $f(0,0) = 0$ and $g(0,0) = 0$, so $(0,0)$ is a critical point. The linearization at $(0,0)$ is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

This is a triangular matrix. It's eigenvalues are 1 and -2.

Real roots, one positive and one negative, means the critical point is a linearized saddle. A saddle is dynamically unstable.

For more accurate sketching, we find eigenvectors: It is easy to compute that $\lambda = 1$ has an eigenvector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\lambda = -2$ has eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



Note: Since the linearized saddle is *structurally stable*, the nonlinear system also looks like a saddle near the critical point. This means our sketch is qualitatively correct near the critical point.

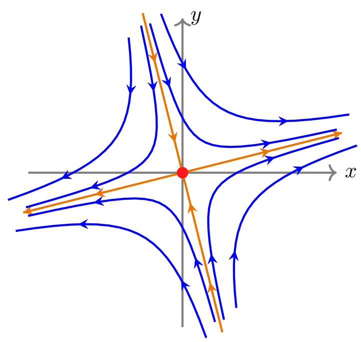
28.7. Solution: Let $f(x,y) = 2x + y + xy^3$ and $g(x,y) = x - 2y - xy$. Clearly $f(0,0) = 0$ and $g(0,0) = 0$, so $(0,0)$ is a critical point. The linearization at $(0,0)$ is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The characteristic equation is $\lambda^2 - 5 = 0$. This has roots $\pm\sqrt{5}$.

Real roots, one positive and one negative, means the critical point is a linearized saddle. A saddle is dynamically unstable.

For more accurate sketching we find eigenvectors: It is easy to compute that $\lambda = \sqrt{5}$ has an eigenvector $\begin{bmatrix} 1 \\ -2+\sqrt{5} \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$ and $\lambda = -\sqrt{5}$ has eigenvector $\begin{bmatrix} 2 \\ -2-\sqrt{5} \end{bmatrix} \approx \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.



Note: Since the linearized saddle is *structurally stable*, the nonlinear system also looks like a saddle near the critical point. This means our sketch is qualitatively correct near the critical point.

28.8. Solution: As usual, let $f(x,y) = 1 - y$ and $g(x,y) = x^2 - y^2$.

(i) The critical points are the solutions to

$$f(x,y) = 1 - y = 0$$
$$g(x,y) = x^2 - y^2 = 0.$$

The top equation gives $y = 1$. Putting this into the bottom equation shows $x = \pm 1$. The critical points are $(1,1)$, $(-1,1)$.

(ii) We linearize around each critical point. The Jacobian is $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix}$.
We examine each critical point in turn.

Critical point (1,1):

$J(1,1) = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}$.

Characteristic equation: $\lambda^2 + 2\lambda + 2 = 0$. Eigenvalues $\lambda = -1 \pm i$.

This is a linearized spiral sink. Looking at the point $(u,v) = (1,0)$, we compute the tangent vector

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Since this points upward, the spiral is counterclockwise.

Critical point (-1,1):

$J(-1,1) = \begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}$.

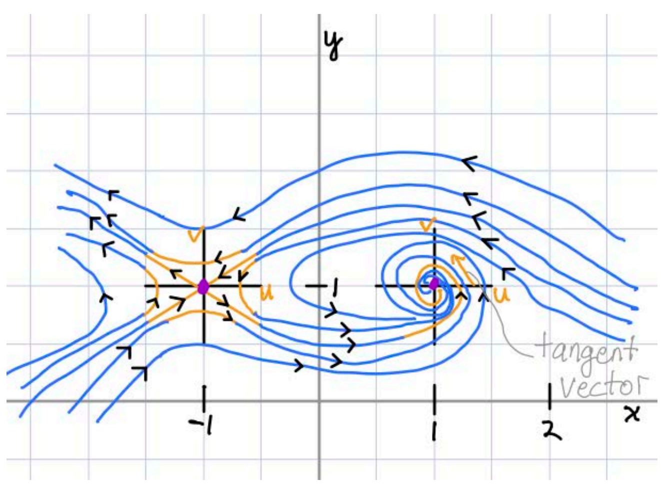
Characteristic equation: $\lambda^2 + 2\lambda - 2 = 0$. Eigenvalues $\lambda = -1 \pm \sqrt{3}$.

This is a linearized saddle. In order to sketch the trajectories near the critical point, we also find the eigenvectors. (We don't show the calculation.)

$\lambda = -1 + \sqrt{3}$ (positive): Eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 - \sqrt{3} \end{bmatrix}$.

$\lambda = -1 - \sqrt{3}$ (negative): Eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 + \sqrt{3} \end{bmatrix}$.

(iii) Here is the sketch. At each critical point the axes are labeled as u and v. Some trajectories near the critical points are drawn in orange. We also show the tangent vector used to determine the sense of the spirals. These are 'tied' together with trajectories drawn in blue.



Note: Since each of the critical points is *structurally stable*, the nonlinear system is qualitatively like the linearized ones near the critical points.

28.9. Solution: (i) Critical points:

$f(x,y) = x - x^2 - xy = x(1-x-y) = 0 \Rightarrow x = 0$ or $1 - x - y = 0$.

$g(x,y) = 3y - xy - 2y^2 = y(3-x-2y) = 0$.

The top equation implies $x = 0$ or $1 - x - y = 0$, i.e., $x = 1 - y$

First assume $x = 0$. Substituting this into the bottom equation: $y(3-2y) = 0$. So, $y = 0$ or $y = 3/2$. We have found two critical points: $(0,0)$ and $(0,3/2)$.

Next, assume $1 - x - y = 0$. Combining this with the bottom equation gives two more critical points: $(1,0)$ and $(-1,2)$.

(ii) We linearize around each critical point. The Jacobian is $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 1-2x-y & -x \\ -y & 3-x-4y \end{bmatrix}$. We examine each critical point in turn.

Critical point (0,0): $J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. This is diagonal, so the eigenvalues are 1, 3. This is a linearized nodal source.

Critical point $(0,3/2)$: $J(0,3/2) = \begin{bmatrix} -1/2 & 0 \\ -3/2 & -3 \end{bmatrix}$. This is triangular, so the eigenvalues are $-1/2$ and -3. This is a linearized nodal sink.

Critical point $(1,0)$: $J(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$. This is triangular, so the eigenvalues are -1 and 2. This is a linearized saddle. To help sketching we find the eigenvectors. (We don't show the arithmetic.)

$\lambda = -1$: Can take $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 2$: Can take $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

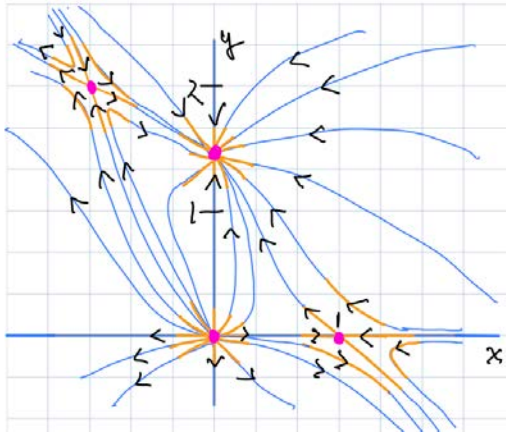
Critical point $(-1,2)$: $J(-1,2) = \begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix}$. This has characteristic equation is $\lambda^2 + 3\lambda - 2 = 0$. The eigenvalues are $\lambda = \dfrac{-3 \pm \sqrt{17}}{2}$.

This is a linearized saddle. To help sketching we find the eigenvectors. (We don't show the arithmetic.)

$\lambda = (-3+\sqrt{17})/2$: Can take $\mathbf{v} = \begin{bmatrix} 1 \\ (-5+\sqrt{17})/2 \end{bmatrix}$

$\lambda = (-3-\sqrt{17})/2$: Can take $\mathbf{v} = \begin{bmatrix} 1 \\ (-5-\sqrt{17})/2 \end{bmatrix}$

(iii) Here is the sketch. Near the linearized nodes we just show trajectories heading towards the critical point (sink) or away from it (source). The trajectories near the critical points are drawn in orange. These are 'tied' together with trajectories drawn in blue.



Note: Since each of the critical points is *structurally stable*, the nonlinear system is qualitatively like the linearized ones near the critical points.

Topic 29. Structural stability.

Solutions

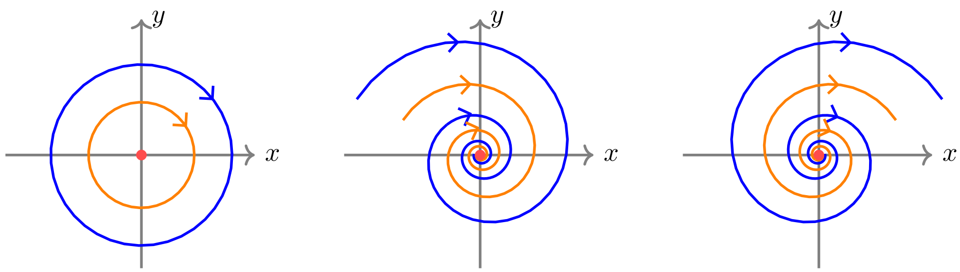
29.1. (a) Solution: The Jacobian for this system is $J(x,y) = \begin{bmatrix} -1-y^2 & 4-2xy \\ -2+2xy & 1+x^2 \end{bmatrix}$.
So, $J(0,0) = \begin{bmatrix} -1 & 4 \\ -2 & 1 \end{bmatrix}$

The characteristic equation is $\lambda^2 + 7 = 0$, so $\lambda = \pm\sqrt{7}i$.

This is a linearized center (pure imaginary eigenvalues). As an equilibrium a center is an edge case in terms of dynamic stability. As a system it is not structurally stable.

The nonlinear system could have a center, spiral sink or spiral source at the origin. Since one of these is a dynamically stable equilibrium and one is unstable, we don't know if the nonlinear equilibrium is dynamically stable or not.

Here are possible phase portraits near (0,0). The direction of rotation is clockwise –because the lower left entry in $J(0,0)$ is negative.



Possible nonlinear phase portraits near a linearized center.

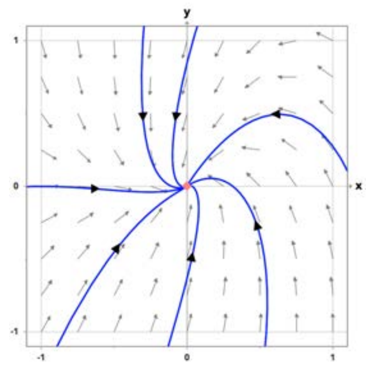
(b) Solution: The Jacobian for this system is $J(x,y) = \begin{bmatrix} -2+2x & -1 \\ 1+3y+2x & -4+3x \end{bmatrix}$. So, $J(0,0) = \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix}$.

The characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$, so $\lambda = -3, -3$.

This is a linearized defective nodal sink (repeated negative eigenvalues). As an equilibrium a defective nodal sink is dynamically stable. As a system it is not structurally stable.

In principle, the nonlinear system could have a defective nodal sink, nodal sink or spiral sink at the origin. Since all of these are dynamically stable equilibria, the nonlinear equilibrium at (0,0) is dynamically stable.

Because the linearization is only valid close to the equilibrium, it doesn't really make sense to worry about which type we have. We draw just one portrait with a sink near (0,0).



Nonlinear phase near linearized defective nodal sink.

29.2. (a) Solution: First we find the critical points. The equations are

$$x' = y = 0$$
$$y' = x(1-x) = 0$$

These are straightforward to solve. The critical points are $(0,0)$ and $(1,0)$.

The Jabobian is $\begin{bmatrix} 0 & 1 \\ 1-2x & 0 \end{bmatrix}$

At $(0,0)$: $J(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The characteristic equation is $\lambda^2 - 1 = 0$. So, $\lambda = \pm 1$.

This is a linearized saddle (one positive, one negative eigenvalue). It is a structurally stable system. That is, the nonlinear system is qualitatively a saddle near $(0,0)$.

To aid in sketching we find the eigenvectors. (We leave the computation to the reader.)

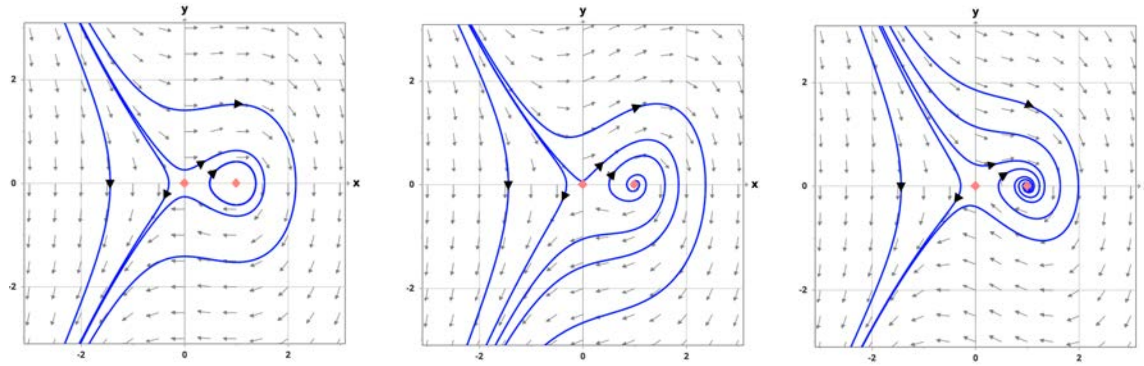
For $\lambda = 1$ we can take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $\lambda = -1$, take $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

At $(1,0)$: $J(1,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

The characteristic equation is $\lambda^2 + 1 = 0$. So, $\lambda = \pm i$.

This is a linearized center (pure imaginary roots). It is not structurally stable. The nonlinear system could have a center, spiral sink or spiral source at the $(1,0)$.

Here are three possible phase portraits for this system. The origin is always a saddle. The critical point at $(1,0)$ is alternately a center, spiral source, spiral sink.



Three possible phase portraits.

Notes. For this system we can show (with more work) that the critical point at $(1,0)$ is a nonlinear center.

To draw the above portraits we actually used the system $x' = y$, $\;y' = x(1-x) + a * \text{sign}(y) * |y|^{1.1}$, with $a = 0, 0.36, -0.36$. When $a = 0$, this system is identical to the one in the problem. For other a, it has the exact same critical points and linearizations as the original systems. A nicer system with the same properties uses $y' = x(1-x) + ay^3$. But this system is a numerically a little harder to work with.

(b) **Solution:** First we find the critical points. The equations are

$$x' = x^2 - x + y = 0$$
$$y' = -yx^2 - y = -y(x^2+1) = 0$$

The second equation implies $y = 0$. Putting this into the first equation, we get $x = 0, 1$. So the critical points are $(0,0)$ and $(1,0)$.

The Jabobian is $\begin{bmatrix} 2x-1 & 1 \\ -2xy & -x^2-1 \end{bmatrix}$

At $(0,0)$: $J(0,0) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$.

This has repeated negative roots, $\lambda = -1, -1$.

The critical point is a linearized defective nodal source. It is not structurally stable. The nonlinear system could be some type of nodal sink or a spiral sink at the origin. The good news is that we are sure it's a sink.

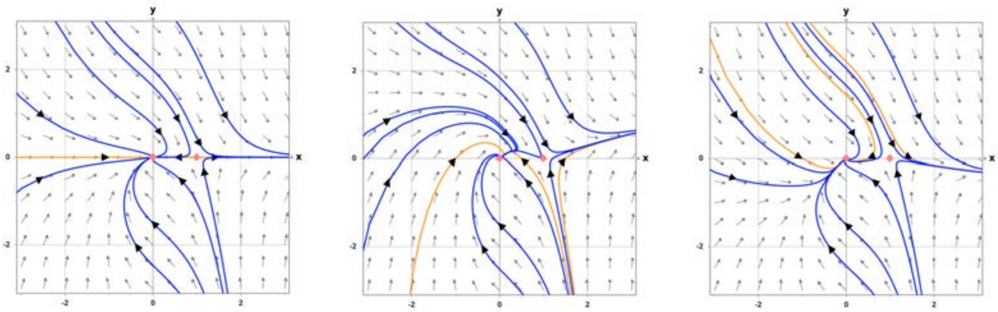
At $(1,0)$: $J(1,0) = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$.

This has eigenvalues $\lambda = 1, -2$. So it is a linearized saddle. Since a saddle is structurally stable, the nonlinear system is qualitatively a saddle near $(1,0)$.

To aid in sketching we find the eigenvectors. (We leave the computation to the reader.)

For $\lambda = 1$ we can take $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. For $\lambda = -2$, take $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Here are three possible phase portraits for this system. Two have nodal sinks at the origin. The third has a spiral sink. The critical point at $(1,0)$ is always a saddle.



Three possible phase portraits.

Notes. For this system we can show that the critical point at $(0,0)$ is a nonlinear node. To see this, note that when $y = 0$, we get trajectories that stay on the x-axis. Since trajectories can't cross, there is no way for another trajectory to spiral around the origin.

To draw the above portraits, we actually used the system $x' = x^2 - x + y, \quad y' = x^2 - x + y + a(x^2(x-1)^2)^{0.55}$, with $a = 0, 1.0, -0.5$. When $a = 0$, this system is identical to the one in the problem. For the other values of a, it has the exact same critical points and linearizations as the original system. A nicer looking system with the same properties uses $y' = x^2 - x + y + ax^2(x-1)^2$. But this system is a numerically a little harder to work with.

Topic 30. Systems: population models

Solutions

30.1. Solution: The new system is

$$x' = \frac{5}{4}ax - pxy$$
$$y' = -by + qxy,$$

The critical point for the old system is

$$\left(\frac{b}{q}, \frac{a}{p}\right).$$

The critical point for the new system is

$$\left(\frac{b}{q}, \frac{5a/4}{p}\right).$$

The effect of fertilizer is to leave the equilibrium flower population the same, but to increase the borer population by 25%. It does not seem like a good idea!

30.2. Solution: Original equations:

$$\text{sharks: } x' = ax - pxy$$
$$\text{fish: } y' = -by + qxy.$$

The original equilibrium is (sharks, fish) $= (\frac{b}{q}, \frac{a}{p})$.

With warming:

$$x' = (a - 0.1)x - pxy$$
$$y' = -(b + 0.1)y + qxy$$

The new equilibrium is (sharks, fish) $= \left(\frac{b+0.1}{q}, \frac{a-0.1}{p}\right)$. So the equilibrium level of sharks increases. (And that of fish decreases.)

30.3. Solution: The Jacobian of the system is $J(x,y) = \begin{bmatrix} 39 - 6x - 3y & -3x \\ -4y & 28 - 2y - 4x \end{bmatrix}$.

(a) $J(0,0) = \begin{bmatrix} 39 & 0 \\ 0 & 28 \end{bmatrix}$. This is a diagonal matrix, so the eigenvalues are the diagonal entries: $\lambda = 39, 28$. Positive real eigenvalues imply the linearized critical point is a nodal source. This is structurally stable, so the nonlinear critical point is also a nodal source.

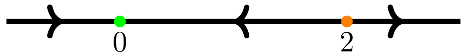
(b) $J(13,0) = \begin{bmatrix} -39 & -39 \\ 0 & -24 \end{bmatrix}$. This is triangular, so the eigenvalues are just the diagonal entries: $\lambda = -39, -24$. Negative eigenvalues imply the linearized critical point is a nodal sink. This is structurally stable, so the nonlinear critical point is also a nodal sink.

(c) $J(0,28) = \begin{bmatrix} -45 & 0 \\ -112 & -28 \end{bmatrix}$. This is triangular, so the eigenvalues are just the diagonal entries: $\lambda = -45, -28$. Negative eigenvalues imply the linearized critical point is a nodal sink. This is structurally stable, so the nonlinear critical point is also a nodal sink.

(d) $J(5,8) = \begin{bmatrix} -15 & -15 \\ -32 & -8 \end{bmatrix}$. The characteristic equation is $\lambda^2 + 23\lambda - 360 = 0$. This has eigenvalues $\dfrac{-23 \pm \sqrt{23^2 + 4 \cdot 360}}{2}$. That is, it has one positive and one negative eigenvalue. Therefore, the linearized critical point is a saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

Note: We could also have identified this as a saddle because its determinant is negative.

30.4. Solution: (a) If $y(t) = 0$, then $x' = x^2 - 2x$. This has critical points $x = 0, 2$ and phase line



So, without any predator ($y(t) = 0$), the prey population x will either crash to 0 or boom to infinity –at least according to this model.

The answer is the same for $y(t)$ if $x(t) = 0$.

(b) Again we can factor to find the critical points.

$$x' = x(x - 2 - y) = 0 \Rightarrow x = 0 \text{ or } x - 2 - y = 0$$
$$y' = y(y - 4 + x) = 0 \Rightarrow y = 0 \text{ or } y - 4 - x = 0.$$

First let $x = 0$, then $y = 0$ or $y = 4$: two critical points (0,0), (0,4).

Next let $y = 0$, then $x = 0$ or $x = 2$: one more critical point (2,0).

Finally, solve $x - 2 - y = 0$, $y - 4 - x = 0$: one more critical (3,1).

The Jacobian is $J(x,y) = \begin{bmatrix} 2x - 2 - y & -x \\ y & 2y - 4 + x \end{bmatrix}$. Looking at each critical point in turn we get

$J(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \Rightarrow \lambda = -2, -4$. Negative eigenvalues imply this is a linearized nodal sink. This is structurally stable so the nonlinear critical point is also a nodal sink.

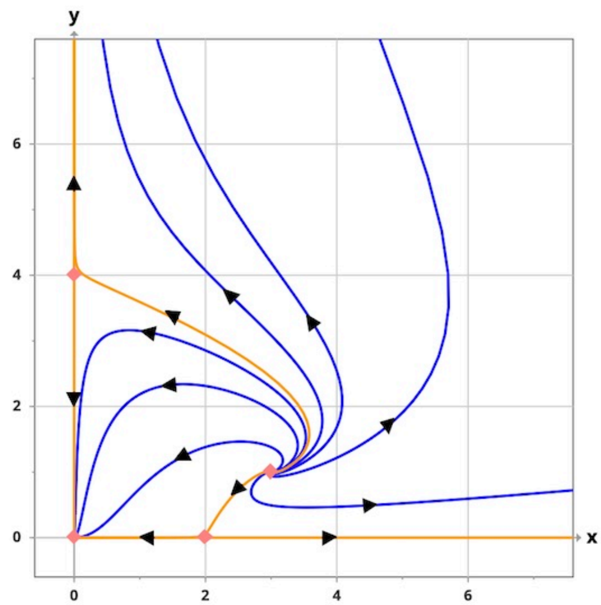
$J(0,4) = \begin{bmatrix} -6 & 0 \\ 4 & 4 \end{bmatrix} \Rightarrow \lambda = -6, 4$. One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable so the nonlinear critical point is also a saddle.

$J(2,0) = \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix} \Rightarrow \lambda = 2, -2$. One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable so the nonlinear critical point is also a saddle.

$J(3,1) = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$.

Characteristic equation: $\lambda^2 - 4\lambda + 6 = 0 \Rightarrow \lambda = 2 \pm \sqrt{2}\, i$ Complex eigenvalues with positive real part imply this is a linearized spiral source. This is structurally stable so the nonlinear critical point is also a spiral source.

(c) Here is the phase portrait. Since we're talking about populations, the portrait only shows the first quadrant.



What seems important, is that each population by itself is modeled by a doomsday-extinction equation. That is, either the population goes to ∞ or to 0. It's hard to tell exactly, but it seems that when the predator (y) goes to infinity, the prey (x) goes extinct.

30.5. Solution: (a) In the presence of y, the growth rate of x decreases. In the presence of x, the growth rate of y increases. Thus x is the prey population and y the predator population.

(b) Without prey, i.e., when $x = 0$, the DE for y is $y' = -y$. This is exponential decay. So eventually the predator population would go to 0.

Without predators, the equation for the prey becomes $x' = 4x - x^2$. This is the logistic equation with dynamically stable critical point $x = 4$ and dynamically unstable critical point $x = 0$. The prey population would eventially stabilize at 4.

(c) We can factor each of the equations to find the critical points:

$$\begin{aligned} x' &= x(4 - x - y) = 0 \;\Rightarrow\; x = 0 \text{ or } 4 - x - y = 0 \\ y' &= y(-1 + x) \;\Rightarrow\; y = 0 \text{ or } x = 1. \end{aligned}$$

The critical points are $(0,0)$, $(4,0)$, $(1,3)$.

The Jacobian is $J(x,y) = \begin{bmatrix} 4 - 2x - y & -x \\ y & -1 + x \end{bmatrix}$.

Considering each of the critical points in turn:

$J(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \lambda = 4, -1.$

One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(4,0) = \begin{bmatrix} -4 & -4 \\ 0 & 3 \end{bmatrix} \Rightarrow \lambda = -4,\, 3.$

One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(1,3) = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}.$

Characteristic equation: $\lambda^2 + \lambda + 3 = 0 \quad \Rightarrow \lambda = -1 \pm \sqrt{11}\, i.$

Complex eigenvalues with negative real part imply this is a linearized spiral sink. This is structurally stable, so the nonlinear critical point is also a spiral sink.

(d) Here is the phase portrait. Since we're talking about populations, the portrait only shows the first quadrant. All trajectories spiral into the critical point at (2,3). (Actually, there are a handful of trajectories along the axes that go asymptotically to the saddle points.)