ES.1803 Part I Problems

Topic 1. Introduction to DEs; modeling; separable equations

1.1. A country is suffering from an epidemic of a contagious disease. Let the total population be P. Assume the rate of change of the number x of people infected is proportional to the product of the number who have the disease and the number who do not.

Write a differential equation modeling this scenario.

1.2. I put my root beer, which is at 20°C, into the freezer, which is at 0°C. After 30 minutes the root beer is at 10°C. How much time (measured from when I first put it into the freezer) will it take before it is at 4°C.

1.3. Write down an explicit solution (involving a definite integral) to the following initial-value problem (IVP):

$$y' = \frac{1}{y^2 \ln x}, \quad y(2) = 0.$$

1.4. Solve the IVP (initial-value problem): $y' = \dfrac{xy + x}{y}, \quad y(2) = 0.$

1.5. Find the general solution by separation of variables: $x\dfrac{dv}{dx} = \sqrt{1 - v^2}$.

1.6. Find all curves $y = y(x)$ whose graphs have the following geometric property. (Use the geometric property to find an ODE satisfied by $y(x)$, and then solve it.)

For each tangent line to the curve, the segment of the tangent line lying in the first quadrant is bisected by the point of tangency.

Topic 2. Linear systems: input-response models

2.1. Find the general solution to $xy' + 2y = x$.

2.2. Consider the ODE $\dfrac{dx}{dt} + ax = r(t)$, where a is a positive constant, $r(t) > 0$ and $\lim_{t\to\infty} r(t) = 0$.

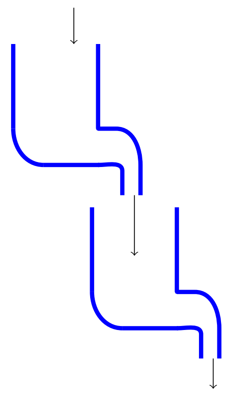
Show that if $x(t)$ is any solution, then $\lim_{t\to\infty} x(t) = 0$. (Hint: use L'Hospital's rule.)

2.3. **Heat transfer.** According to Newton's law of cooling, the rate at which the temperature T of a body changes is proportional to the difference between T and the external temperature.

At time $t = 0$, a pot of boiling water is removed from the stove. After five minutes, the water temperature is $80^o C$. If the room temperature is $20^o C$, when will the water have cooled to $60^o C$? (Set up and solve an ODE for $T(t)$.)

Topic 3. Input-response models continued

3.1. Consider the cascade of two mixing tanks shown. The volume of salt solution in each tank is 100 liters. The top tank initially has 30kg of salt and the bottom has 15kg. The flow rates into and out of each tank are 5 liters/minute, with pure water flowing into the top tank.



Let $x(t)$ be the amount of salt in the top tank and let $y(t)$ be the amount in the bottom tank.

(a) Write a DE for $x(t)$ and solve it.

(b) Show that $\dfrac{dy}{dt} = \dfrac{5x}{100} - \dfrac{5y}{100}$ and solve for $y(t)$.

Topic 4. Complex numbers and exponentials

4.1. Express $\dfrac{1-i}{1+i}$ in the form $a+bi$ using two methods: First, by changing the numerator and denominator to polar form. Second using rectangular coordinates throughout. Show that the two answers agree.

4.2. Calculate the following two ways: first by changing to polar form, and second using the binomial theorem.

(a) $(1-i)^4$

(b) $(1+i\sqrt{3})^3$

4.3. By using Euler's formula and the binomial theorem, express $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$

4.4. (a) Solve the equation $x^4 + 16 = 0$.

(b) Solve the equation $x^4 + 2x^2 + 4 = 0$, expressing the four roots in both polar and rectangular form.

4.5. Find $I = \displaystyle\int e^{3x} \sin(4x)\, dx$ using complex replacement.

Topic 5. Homogeneous, linear, constant coefficient DEs

5.1. Find the general real-valued solution to each of the following.

(a) $y'' - 3y' + 2y = 0$.

(b) $y'' + 2y' + 2y = 0$. (Give the solution in both polar and rectangular form.)

(c) $y'' - 2y' + 5y = 0$; $y(0) = 1$, $y'(0) = -1$

5.2. Find the general real-valued solution to each of the following.

(a) $y^{(6)} - y = 0$.

(b) $y^{(4)} + 16y = 0$.

Topic 6. Operators, ERF and SRF

Note: the Exponential Response Formula is also called the Exponential Input Theorem.

6.1. (a) Let $P(D) = D^2 + 6D + 12I$. Find the general solution to $P(D)y = e^{2t}$.

(b) Find a particular solution to $y'' + 4y' + 12y = e^{-2t}$.

6.2. Let $P(D) = D^2 + 7D + 12I$.

(a) Find the general solution to $P(D)y = \cos(2t)$

(b) Find a particular solution to $P(D)y = \sin(2t)$

(c) Find the general solution to $P(D)y = e^{2t}\cos(3t)$

6.3. Find a particular solution to $y'' + 7y' + 12y = e^{-4t}$

6.4. (a) Find a particular solution to $y'' + 9y = \cos(t)$

(b) Find a particular solution to $y'' + 9y = \cos(3t)$

6.5. Find the general real-valued solution to $y^{(4)} + 2y'' + 4y = \cos(3t)$

Topic 7. Inhomegenous DEs; UC methods; theory

7.1. Find particular solutions to each of the following differential equations.

(a) $y'' - y' + 3y = 3t + 5$

(b) $y'' + 8y' + 7y = t^2$

(c) $y^{(5)} + 3y^{(4)} + 2y = 21$

7.2. Theory: Existence and Uniqueness Theorem.

(a) By differentiating, find a second-order linear homgeneous DE of the form

$$y'' + p(t)y' + q(t)y = 0$$

that has $y(t) = t^2$ as a solution.

(b) The solutions

$$y(t) = t^2 \text{ and } y(t) = 0$$

both satisfy the same initial conditions at $t = 0$, i.e., $y(0) = 0$ and $y'(0) = 0$. Why is this not contradicted by the Existence and Uniqueness Theorem from the Topic 7 class notes?

Topic 8. Applications: stability

8.1. Consider $y'' + by' + 4y = 0$, where b is a constant. For each statement below, tell for what value(s) of b it holds (indicate reasoning):

(a) the equation has oscillatory solutions

(b) all solutions are damped oscillations

8.2. The equation $mx'' + bx' + kx = 0$ represents the motion of a damped spring-mass system. (The independent variable is the time t.)

How are the constants m, b, k related if the system is critically damped (i.e., just on the edge of being oscillatory)?

8.3. A series RLC-circuit is modeled by either of the ODEs (the second equation is just the derivative of the first)

$$Lq'' + Rq' + \frac{q}{C} = E,$$
$$Li'' + Ri' + \frac{i}{C} = E',$$

where $q(t)$ is the charge on the capacitor, $i(t)$ is the current in the circuit and $E(t)$ is the applied electromotive force (from a battery or generator). The constants L, R, C are respectively the inductance of the coil, the resistance, and the capacitance, measured in some compatible system of units.

(a) Show that if $R = 0$ and $E = 0$, then $q(t)$ varies periodically, and find the period. (Assume $L \neq 0$.)

(b) Assume $E = 0$; how must R, L, C be related if the current oscillates?

(c) If $R = 0$ and $E = E_0 \sin \omega t$, then for a certain ω_0, the current will have large amplitude whenever $\omega \approx \omega_0$. What is the value of ω_0. (Indicate reason.)

8.4. Consider the system $y'' + 2y' + cy = 0, \quad c$ a constant .

(a) Different values of c give different types of characteristic roots, e.g., both positive, one positive one negative, etc. For each type give the range of values of c which give that type.

(b) Summarize the above information in Part (a) in a c-axis, and marking the intervals on it corresponding to the different possibilities for the roots of the characteristic equation.

Finally, use this information to mark the interval on the c-axis for which the corresponding ODE is stable.

Extra material on non-constant coefficient linear equations.

Note. In ES.1803, we used to do a little work with second-order nonconstant coefficient DEs. The following two problems are about such equations. To solve them, you will need the following formulas.

1. Wronskian of two functions: For functions $y_1(x)$ and $y_2(x)$, their Wronskian is

$$W(x) = \det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix}.$$

2. Variation of parameters formula: Consider the inhomogeneous second-order linear DE and its associated homogeneous equation.

$$y'' + p(x)y' + q(x)y = f(x) \qquad \text{((I) Inhomogeneous)}$$
$$y'' + p(x)y' + q(x)y = 0 \qquad \text{((H) Homogeneous)}$$

If y_1 and y_2 are basic solutions to (H), then the solution to (I) is given by

$$y(x) = -y_1(x)\left(\int \frac{y_2(x)}{W(x)} f(x)\,dx + C_1\right) + y_2(x)\left(\int \frac{y_1(x)}{W(x)} f\,dx + C_2\right).$$

Here, $W(x)$ is the Wronskian of y_1, y_2.

8.5. Find a particular solution by variation of parameters.

(a) $y'' + y = \tan(x)$.

(b) $y'' + 2y' - 3y = e^{-x}$.

(c) $y'' + 4y = \sec^2(2x)$.

8.6. **Bessel's equation of order** p is $x^2y'' + xy' + (x^2 - p^2)y = 0$. For $p = 1/2$ two indendent solutions for $x > 0$ are

$$y_1(x) = \frac{\sin(x)}{\sqrt{x}} \quad \text{and} \quad y_2(x) = \frac{\cos(x)}{\sqrt{x}}, \quad x > 0.$$

Use this to find the general solution to $x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{3/2}\cos(x)$.

Topic 9. Applications: frequency response

9.1. For each of the following systems, with input $f(t)$ and output $x(t)$ use your calculus graphing skills to plot the graph of the amplitude response (i.e., gain vs. ω). If there is a practical resonant frequency say what it is.

(a) $x'' + x' + 7x = f(t)$, where $f(t) = F_0 \cos(\omega t)$.

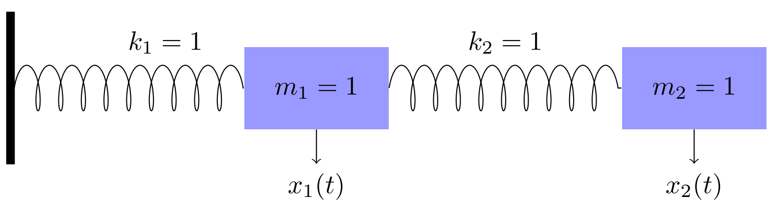
(b) $x'' + 8x' + 7x = f(t)$, where $f(t) = F_0 \cos(\omega t)$.

9.2. Do problem 3c from the Topic 8 Part I problems.

9.3. Coupled springs. A system consisting of two coupled springs is modeled by the pair of DEs.

$$x_1'' + 2x_1 - x_2 = 0; \qquad x_2'' + x_2 - x_1 = 0.$$

Here x_1 is the displacement of mass m_1 from equilibrium and x_2 is the displacement of m_2. For this problem we took both masses and both spring constants to be 1 (in compatible units).



(a) Eliminate x_1 from these equations to get a fourth-order DE for x_2.

(b) Solve the DE from Part (a) to find the general solution for x_2.

Topic 10. Direction fields, integral curves, existence of solutions

10.1. For each of the following ordinary differential equations, draw a direction field by using about five isoclines; the picture should be square, using the intervals between -4 and 4 on both axes. Then, sketch in some integral curves, using the information provided by the direction field.

Finally, do whatever else is asked.

(a) $y' = -\frac{y}{x}$. Also, solve the equation exactly and compare your integral curves with the correct ones.

(b) $y' = 2x + y$. Also, find a solution whose graph is also an isocline, and verify this fact analytically (i.e., by calculation, not from a picture). (Remember: this is not to be expected in most cases!)

(c) $y' = x^2 + y^2 - 1$.

(d) $y' = \frac{1}{x+y}$. Be sure to include the isoclines for $m = 1$ and '$m = \infty$'.

You should see that the curve $y = -x - 1$ is a solution.

Sketch in some curves that follow the directection field. The curves are very pretty, but we won't call these integrals curves because those that pass through the $m = \infty$ isocline have two y values for each x value. That is, they are not the graph of a function $y = y(x)$.

Each of these curves represent two solutions both starting on the $m = \infty$ isocline. That is, they are a combination of two integral curves.

Will any other integral curves cross the line $y = -x - 1$? Explain by using the existence and uniqueness theorem

10.2. Consider the differential equation $y' = \frac{-y}{x^2 + y^2}$.

(a) Sketch a direction field with several isoclines, including the nullcline, for the ODE

Explain, using it and the ODE itself how one can tell that the solution $y(x)$ satisfying the initial condition $y(0) = 1$ is a decreasing function for $y > 0$ and is always positive for $x > 0$
.

(b) Redo the sketch. This time, just draw the nullclines and then label each region in the plane with a plus or a minus depending on the slope of the direction field in that region. Using just this information, sketch in a few solutions.

10.3. According to the existence and uniqueness theorem, under what conditions on $a(x), b(x)$, and $c(x)$ will the initial value problem (IVP)

$$a(x)\, y' + b(x)\, y = c(x), \quad y(x_0) = y_0$$

be guaranteed to have a unique solution in some interval $[x_0 - h, x_0 + h]$ centered around x_0?

Topic 11. Numerical methods for first-order ODEs

11.1. Let $y(x)$ be the solution to the IVP $y' = x - y; \quad y(0) = 1.$

(a) Use the Euler method with step size $h = 0.1$ to find an approximate value of $y(x)$ for $x = 0.1, 0.2, 0.3$

Is your estimate for $y(0.3)$ too high or too low, and why?

(b) Use the RK2 method (also called Modified Euler, Improved Euler, or Heun's method) and the step size $h = 0.1$ to determine the approximate value of $y(0.1)$.

Did the estimate in Part (b) move the estimate in Part (a) in the correct direction, e.g., if the estimate in Part (a) was too low, did Part (b) give an increased estimate?

Topic 12. Autonomous DEs and bifurcation diagrams

12.1. For each of the following autonomous equations $dx/dt = f(x)$, do the following:

(i) Draw the phase line. Be sure to label each critical point as stable, unstable or semi-stable.

(ii) Use the information in the phase line to make a second picture showing the tx-plane, with a set of typical solutions to the DE: the sketch should show the main qualitative features (e.g., the constant solutions, asymptotic behavior of the non-constant solutions).

(a) $x' = x^2 + 2x$

(b) $x' = -(x-1)^2$

(c) $x' = 2x - x^2$

(d) $x' = (2-x)^3$

12.2. Consider $y' = y(y-a-1)(y-2a)$, where a is a constant. **(a)** Draw the bifurcation diagram. (y vs. a)

(b) For what values of a is this sustainable?

(c) Identify all the bifurcation points.

Topic 13. Linear algebra: matrices, vector spaces, linearity

13.1. Vectors and matrices

(a) Write a row vector with norm 1 (the square root of the sum of the squares of the entries).

(b) Write a column vector with 4 entries whose entries add to zero.

In Parts (c) and (d), let

$$A = \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(c) Find a vector $\mathbf{v}$ such that $A\mathbf{v}$ is the third column of A.

(d) Find a vector $\mathbf{w}$ such that $\mathbf{w}A$ is the third row of A.

13.2. Which of the following sets are vector spaces?

(a) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbf{R}^3$ such that $x + y + z = 0$.

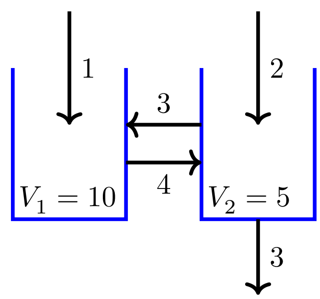
(b) The set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbf{R}^3$ such that $x + y + z = 1$.

(c) The set of functions $f(t)$ of t such that $f(0) = 0$ and $f(\pi) = 0$.

(d) The set of all linear combinations in $\mathbf{R}^8$ of the vectors:

$(1, -2, 0, 0, 0, 0, 0, 0)$, $(0, 1, -2, 0, 0, 0, 0, 0)$, $(0, 0, 1, -2, 0, 0, 0, 0)$, $(0, 0, 0, 1, -2, 0, 0, 0)$

13.3. Consider the two compartment system with flow rates in liters/minute and volumes in liters as shown. Suppose the concentration of solute in the inflows are 3g/l and 2g/l for tanks 1 and 2 respectively.



(a) Give the system of DEs modeling the amounts of solute $x(t)$, $y(t)$ in tanks 1 and 2. Write your answer in matrix form.

(b) Find a particular solution $\mathbf{x_p}$ to this inhomogeneous DE by guessing a **constant** solution.

Topic 14. Linear algebra: row reduction and subspaces

14.1. Find each of the following matrix products, and their ranks.

(a) $\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 0 & -1\end{bmatrix}$ **(b)** $\begin{bmatrix}1 & 2 & -1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$ **(c)** $\begin{bmatrix}1 & 2 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\\2 & 3\end{bmatrix}$

14.2. (a) Which of the following matrices is in reduced row-echelon form?

(i) $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 5\\0 & 0\end{bmatrix}$ (iii) $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ (iv) $\begin{bmatrix}1 & 2 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$ (v) $\begin{bmatrix}0\end{bmatrix}$

(b) Find the reduced echelon form of each of the following matrices.

(i) $\begin{bmatrix}4\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 1\end{bmatrix}$ (iii) $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ (iv) $\begin{bmatrix}-2 & 1 & 0\\1 & -2 & 1\\0 & 1 & -2\end{bmatrix}$

14.3. (Column space, null space, independence, basis, dimension.)

(a) Write a matrix equation that shows that the vectors

$$\mathbf{v_1} = \begin{bmatrix}1\\0\\0\end{bmatrix},\ \mathbf{v_2} = \begin{bmatrix}1\\1\\0\end{bmatrix},\ \mathbf{v_3} = \begin{bmatrix}1\\1\\1\end{bmatrix},\ \mathbf{v_4} = \begin{bmatrix}2\\3\\4\end{bmatrix},$$

are linearly dependent (or, more properly, form a linearly dependent set).

(b) (i) Find a basis for the null spaces of the following matrices. Do this by first finding the reduced echelon form, then, setting each free variable equal to 1 and the others to zero, one at a time.

$$A = \begin{bmatrix}0 & 1 & 2 & 3\\1 & 2 & 3 & 4\\2 & 3 & 4 & 5\end{bmatrix},\ B = \begin{bmatrix}0 & 1 & 2\\1 & 2 & 3\\2 & 3 & 4\\3 & 4 & 5\end{bmatrix}$$

(Note: $B = A^T$.)

(ii) Write down the general solution to $A\mathbf{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

(iii) Write down the general solution to $B\mathbf{y} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$.

(c) Find a basis for each of the following subspaces of $\mathbf{R}^4$. Do this in (ii) and (iii) by expressing the subspace as the null space of an appropriate matrix, and finding a basis for that null space by finding the reduced echelon form. In each case, state the dimension of this subspace.

(i) All vectors whose entries are all the same.

(ii) All vectors whose entries add to zero.

(iii) All vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $x_1 + x_2 = 0$ and $x_1 + x_3 + x_4 = 0$.

(d) (i) For which numbers c and d does the column space of the matrix

$$\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

have dimension 2?

(ii) Find numbers c and d such that the null space of the matrix

$$\begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$$

is 3-dimensional.

Topic 15. Linear algebra: transpose, inverse, determinant

15.1. (Determinants and Inverses.)
Summary of properties of the determinant

(0) $\det A$ is only defined for square matrices.
(1) $\det I = 1$.
(2) Adding a multiple of one row to another does not change the determinant.
(3) Multiplying a row by a number c multiplies the determinant by c.
(4) If you swap two rows, you reverse the sign of the determinant. (5) $\det(AB) = \det(A)\det(B)$.
(6) A is invertible exactly when $\det A \neq 0$.

Compute the determinants of the following matrices, and if the determinant is nonzero find the inverse.

(a) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

15.2. (Rotation matrices.)
Let $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(a) Compute $R(\alpha)R(\beta)$. Show that it is $R(\gamma)$ for some angle γ.

(b) Compute $\det R(\theta)$ and $R(\theta)^{-1}$.

Topic 16. Linear algebra: eigenvalues, diagonalization

16.1. (Eigenvalues and Eigenvectors.)

(a) (i) Find the eigenvalues and eigenvectors of the matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix}$.

(ii) Find the eigenvalues of AA and AB (It's not generally the case that the eigenvalues of a product are the products of the eigenvalues.)

(iii) If you know the eigenvalues of A, what can you say about the eigenvalues of cA (where c is some constant, and cA means A with all entries multiplied by c)?

(iv) Find the eigenvalues of $A + B$ for the matrices A and B in (i). (It's not generally the case that the eigenvalues of a sum are the sum of the eigenvalues.)

(b) Find the characteristic polynomial of each of the following matrices.

(i) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

16.2. (Diagonalization, Orthogonal Matrices)

(a) (i) Diagonalize each of the following matrices: that is, find an invertible S and a diagonal Λ such that the matrix factors as $S\Lambda S^{-1}$.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

(ii) Write down diagonalizations of A^3 and A^{-1}.

(b) Suppose that A is a 10×10 matrix of rank 1 and trace 5. What are the ten eigenvalues of A? (Remember, eigenvalues can be repeated! and the trace of a matrix, defined as the sum of its diagonal entries, is equally well the sum of its eigenvalues (taken with repetition).)

(c) A matrix S is *orthogonal* when its columns are orthogonal to each other and all have length (norm) 1. This is the same as saying that $S^T S = I$. Think about why this is true!

Write the symmetric matrix $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ as $S\Lambda S^{-1}$ with Λ diagonal and S orthogonal.

16.3. (Two dimensional linear dynamics.)

(a) Diffusion: A door is open between rooms that initially hold $v(0) = 30$ people and $w(0) = 10$ people. People tend to move to the less crowded room. Let's suppose that the movement is proportional to $v - w$:

$$v' = w - v \quad , \quad w' = v - w$$

(i) Write this system as a matrix equation $\mathbf{u}' = A\mathbf{u}$: What is A?

(ii) Find the eigenvalues and eigenvectors of this matrix.

(iii) What are v and w at $t = 1$

(iv) What are v and w at $t = \infty$? (Some parties last that long!)

(b) Find all the solutions to $\mathbf{u}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{u}$ which trace out the circle of radius 1.

Topic 17. Matrix methods of solving systems of DEs

17.1. (Normal modes)

(a) Suppose A and B are square matrices with eigenvalues $\lambda_1, \ldots \lambda_m$ and $\mu_1 \ldots \mu_n$. What are the eigenvalues of $C = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$?

(b) Suppose A is an $n \times n$ matrix and I is the $n \times n$ identity. Express the eigenvalues of the $2n \times 2n$ matrix $C = \left[\begin{array}{c|c} 0 & I \\ \hline A & 0 \end{array}\right]$ in terms of the eigenvalues of A.

17.2. (Decoupling.)
Farmer Jones and Farmer McGregor have adjacent farms, both afflicted with rabbits. Rabbits breed fast, with a growth rate of 5 per year. Let $x(t)$ be the number of rabbits in Jones' farm and $y(t)$ the number in McGregor's. These systems are coupled: the rabbits can jump over the hedge between the farms. McGregor's grass is greener and the system of equations turns out to be

$$\begin{aligned} x' &= 3x + y \\ y' &= 2x + 4y \end{aligned}$$

In matrix form the equation is

$$vbx' = A\mathbf{x}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

(a) Explain how the growth rate of 5 is reflected in these equations.

(b) The grass on McGregor's farm is greener. How is this reflected in the system of equations?

(c) We'll tell you that the eigenvalues and eigenvectors for this system are

$$\lambda_1 = 5, \, \mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad \lambda_2 = 2, \, \mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Diagonalize the matrix A and decouple this system

17.3. Solve $\mathbf{x}' = A\mathbf{x}$ for each of the following matrices A.

(a) $A = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}$

(b) $B = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$

(Hint: you should find the characteristic polynomial is $(\lambda - 1)(\lambda - 2)(\lambda + 1)$.)

17.4. (Complex eigenvalues) Solve the system $\mathbf{x}' = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \mathbf{x}$. Give the general *real-valued* solution.

17.5. (Complex eigenvalues) Solve the system $\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \mathbf{x}$. Give the general *real-valued* solution.

Topic 18. Matrix exponential, exponential and sinusoidal input

18.1. Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. Solve the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Do this using the matrix exponential. You can leave the matrix exponential as a product of 3 matrices.

18.2. Let $A = \begin{bmatrix} -1 & 1 \\ -5 & 3 \end{bmatrix}$. Use the exponential response formula to do the following.

(a) Find a particular solution to $\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$.

(b) Find a particular solution to $\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} e^{2t} \\ e^{3t} \end{bmatrix}$.

(c) Find a particular solution to $\mathbf{x}' = A\mathbf{x} + \begin{bmatrix} \sin(t) \\ 0 \end{bmatrix}$.

18.3. Find a solution to $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}$.

Topic 19. Fundamental matrix, variation of parameters

19.1. Consider the second-order equation $x'' + p(t)x' + q(t)x = f(t)$. Write the companion system $\mathbf{x}' = A\mathbf{x} + \mathbf{F}$.

19.2. Let $\mathbf{x_1}(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$ and $\mathbf{x_2}(t) = \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$ be two vector functions.

(a) Prove by using the definition that $\mathbf{x_1}$ and $\mathbf{x_2}$ are independent.

(b) Calculate the Wronksian $W(\mathbf{x_1}, \mathbf{x_2})$.

(c) Find a linear system $\mathbf{x}' = A\mathbf{x}$ having $\mathbf{x_1}$ and $\mathbf{x_2}$ as solutions. (Hint, use the relation $\Phi' = A\Phi$ for a fundamental matrix Φ.)

(d) The solution $\mathbf{x_2}(t)$ has $\mathbf{x_2}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The trivial solution to the system in Part (c) also takes this value. Why does this not violate the existence and uniqueness theorem?

19.3. Use variation of parameters to solve $\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^{-2t} \\ -2e^t \end{bmatrix}$.

19.4. Use variation of parameters to solve $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$.

Topic 20. Step and delta functions.

20.1. (Integration) Compute

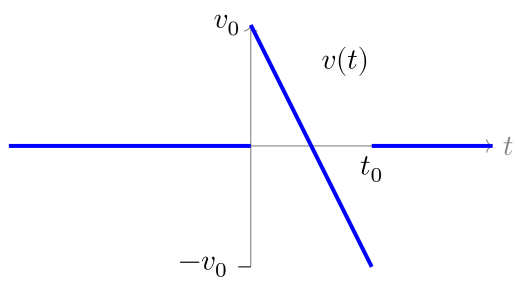
$$\int_{0^-}^{10} 5\delta(t+1) + 3\delta(t) + t^2\delta(t-5) + t\delta(t-20)\, dt.$$

20.2. (Differentiation) When you fire a gun, you exert a very large force on the bullet over a very short period of time. If we integrate $F = ma = mx"$ we see that a large force over a short time creates a sudden change in the momentum, mx'. This is called an "impulse."

If the gun is fired straight up, the graph of the elevation of the bullet, plotted against t, starts at zero, then rises in an inverted parabola, and then when it hits the ground it stops again.

The velocity (derivative of the position function) is zero for $t < 0$; then at $t = 0$ it rises to v_0 (the initial velocity of the bullet); then it falls at constant acceleration (of gravity) until the instant when it hits the ground, when it returns abruptly to zero.

The graph of $v(t)$ is shown at below.



Give a formula for the generalized derivative of $v(t)$ and sketch its graph.

20.3. Solve the following DEs and graph the solution.

(a) $2x'' + 2x' = \delta(t)$ with rest IC.

(b) $2x'' + 2x' = \delta(t-1)$ with rest IC.

Topic 21. Fourier series: basics.

21.1. Find the smallest period for each of the following.

(a) $\sin(\pi t/3)$ **(b)** $|\sin(t)|$ **(c)** $\cos^2(3t)$.

21.2. The function $f(t)$ has period 1. Over the interval $0 < t < 1$ we have $f(t) = t$. Sketch the graph of $f(t)$ over 3 full periods and find the Fourier series for $f(t)$

Topic 22. Fourier series introduction continued.

22.1. The function $f(t)$ has period 1. Over the interval $0 < t < 1$ we have $f(t) = \sin(\pi t)$. Sketch the graph of $f(t)$ over 3 full periods and find the Fourier series for $f(t)$

22.2. Find the Fourier series for the period 2π function which is given over the interval $-\pi < t < \pi$ by

$$f(t) = \begin{cases} -t & \text{for } -\pi < t < 0 \\ t & \text{for } 0 < t < \pi \end{cases}$$

Topic 23. Sine and cosine series; calculation tricks.

23.1.

(a) Find the Fourier sine series of the function $f(x) = 1 - x$ over the interval $0 < x < 1$

(b) Find the Fourier cosine series of the function $f(x) = 1 - x$ over the interval $0 < x < 1$

Topic 24. Linear ODEs with periodic input.

24.1. For each spring-mass system, say whether or not pure resonance occurs without actually finding the solution.

(a) $2x'' + 10x = f(t)$; where $f(t) = 1$ on the interval $(0, 1)$, $f(t)$ is odd, and of period 2.

(b) $x'' + 4\pi^2 x = f(t)$; where $f(t) = 2t$ on the interval $(0, 1)$, $f(t)$ is odd, and of period 2.

(c) $x'' + 9x = f(t)$; where $f(t) = 1$ on the interval $(0, \pi)$, $f(t)$ is odd, and of period 2π.

24.2. Find a periodic solution as a Fourier series to $x'' + x' + 3x = f(t)$, where $f(t) = 2t$ on $(0, \pi)$, $f(t)$ is odd, and has period 2π.

24.3. For the following lightly damped spring-mass systems, determine what term of the Fourier series solution should dominate, i.e., have the biggest amplitude.

(a) $2x'' + 0.1x' + 18x = f(t)$, where $f(t) = 2t$ on $(0, \pi)$, $f(t)$ is odd, and has period 2π.

(b) $3x'' + x' + 30x = f(t)$, where $f(t) = t - t^2$ on $(0, 1)$, $f(t)$ is odd, and has period 2.

Topic 25. PDEs; separation of variables.

25.1. (Heat equation) Solve the following boundary value problem using Fourier's method of separation of variables.

$$\begin{aligned} u_t &= u_{xx}, \text{ on } 0 \le x \le 10, \text{ and } t > 0 \\ u_x(0,t) &= u_x(10,t) = 0 \\ u(x,0) &= 4x. \end{aligned}$$

25.2. (Wave equation) Solve the following boundary value problem using Fourier's method of separation of variables.

$$\begin{aligned} y_{tt} &= 4y_{xx}, \text{ on } 0 \le x \le \pi, \text{ and } t > 0 \\ y(0,t) &= y(\pi,t) = 0 \\ y(x,0) &= \tfrac{1}{10}\sin(2x), \quad y_t(x,0) = 0. \end{aligned}$$

25.3. (Wave equation) Solve the following boundary value problem using Fourier's method of separation of variables.

$$\begin{aligned} y_{tt} &= 100y_{xx}, \text{ on } 0 \le x \le 1, \text{ and } t > 0 \\ y(0,t) &= y(1,t) = 0 \\ y(x,0) &= 0, \quad y_t(x,0) = x. \end{aligned}$$

Topic 27. Qualitative behavior of linear systems.

27.1. Consider the system $x' = -x;\quad y' = -2y$.

(a) Solve the system by inspection or using eigenvalues and eigenvectors. Then sketch the phase portrait of trajectories in the phase plane. Include arrows giving the direction of increasing time on the trajectories.

Give the type and dynamic stability of the critical point at the origin.

(b) How would the phase portrait change if the system were $x' = x;\quad y' = 2y$?

In this case, what is the type and dynamic stability of the critical point at the origin?

27.2. Sketch the phase portrait for each of the following. Also give the type and dynamic stability of the critical point at the origin for each.

(a) $\begin{array}{rcl} x' & = & 2x - 3y \\ y' & = & x - 2y \end{array}.$

(b) $\begin{array}{rcl} x' & = & 2x \\ y' & = & 3x + y \end{array}.$

(c) $\begin{array}{rcl} x' & = & -2x - 2y \\ y' & = & -x - 3y \end{array}.$

(d) $\begin{array}{rcl} x' & = & x - 2y \\ y' & = & x + y \end{array}.$

(e) $\begin{array}{rcl} x' & = & x + y \\ y' & = & -2x - y \end{array}.$

27.3. Consider the damped spring-mass system

$$mx'' + bx' + kx = 0, \quad m, b, k > 0.$$

(a) Write this as an equivalent first-order linear system.

(b) Suppose $b = 0$. What is the type of the critical point at $(0, 0)$? Is it dynamically stable?

(c) Suppose b is small relative to m and k. What is the type of the critical point at $(0, 0)$? Is it dynamically stable? In which sense (clockwise or counterclockwise) do the trajectories rotate?

(d) Suppose b is large relative to m and k. What is the type of the critical point at $(0, 0)$? Is it dynamically stable?

(e) Can the critical point be a saddle?

Topic 28. Qualitative behavior of nonlinear systems.

28.1. Find the critical points of each of the following nonlinear autonomous systems.

(a) $\begin{aligned} x' &= x^2 - y^2 \\ y' &= x - xy. \end{aligned}$

(b) $\begin{aligned} x' &= 1 - x + y \\ y' &= y + 2x^2. \end{aligned}$

28.2. Write each of the following equations as an equivalent first-order system, and find the critical points.

(a) $x'' + a(x^2 - 1)x' + x = 0.$

(b) $x'' - x' + 1 - x^2 = 0.$

28.3. Consider the system

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

In general, what can you say about the relation between the trajectories and the critical points of this system and those of the following related systems?

(a) $\begin{aligned} x' &= -f(x, y) \\ y' &= -g(x, y). \end{aligned}$

(b) $\begin{aligned} x' &= g(x, y) \\ y' &= -f(x, y). \end{aligned}$

28.4. Consider the autonomous system
$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$ with solution $(x(t), y(t))$

(a) Show that $(\tilde{x}(t), \tilde{y}(t)) = (x(t - t_0), y(t - t_0))$ is also a solution. What is the geometric relation between the two solutions, i.e., the geometric relation between their respective trajectories?

(b) The existence and uniqueness theorem for the system says that if f and g are continuously differentiable everywhere, there is one and only one solution $(x(t), y(t))$ satisfying a given initial condition $(x(t_0), y(t_0)) = (a, b)$.

Using this and Part (a), show that two trajectories either trace the same curve or do not intersect anywhere.

(Note that if two trajectories intersect at a point (a, b), the corresponding solutions which trace them out may be at (a, b) at different times. Part (a) shows one way this can happen.)

28.5. For the following system, the origin is clearly a critical point. Give its geometric type and dynamic stability, and sketch some nearby trajectories of the system.

$$\begin{aligned} x' &= x - y + xy \\ y' &= 3x - 2y - xy. \end{aligned}$$

28.6. For the following system, the origin is clearly a critical point. Give its geometric type and dynamic stability, and sketch some nearby trajectories of the system.

$$\begin{aligned} x' &= x + 2x^2 - y^2 \\ y' &= x - 2y + x^3. \end{aligned}$$

28.7. For the following system, the origin is clearly a critical point. Give its geometric type and dynamic stability, and sketch some nearby trajectories of the system.

$$\begin{aligned} x' &= 2x + y + xy^3 \\ y' &= x - 2y - xy. \end{aligned}$$

28.8. For the following system, carry out our program for sketching trajectories. That is, (i) find the critical points, (ii) analyze each and draw in nearby trajectories, (iii) add some other trajectories compatible with the ones you have drawn; when necessary, put in a vector from the vector field to help.

$$\begin{aligned} x' &= 1 - y \\ y' &= x^2 - y^2. \end{aligned}$$

28.9. Repeat problem 8 for the system

$$\begin{aligned} x' &= x - x^2 - xy \\ y' &= 3y - xy - 2y^2. \end{aligned}$$

Topic 29. Structural stability.

29.1. Each of the following systems has a critical point at the origin. For this critical point, find the geometric type and dynamic stability of the corresponding linearized system. Say whether the system is structurally stable near the critical point and then tell what the possibilities would be for the corresponding critical point of the given nonlinear system.

Is the nonlinear equilibrium at the origin dynamically stable?

(a) $x' = -x + 4y - xy^2, \quad y' = -2x + y + x^2 y$

(b) $x' = -2x - y + x^2, \quad y' = x - 4y + 3xy + x^2$

29.2. Each of the following systems has one critical point whose linearization is not structurally stable. In each case, sketch several pictures showing the different ways the trajectories of the nonlinear system might look. (We want you to draw the possible phase portraits including all the critical points.)

Begin by finding the critical points and determining the type of the corresponding linearized system at each of the critical points.

(a) $x' = y, \quad y' = x(1 - x).$

(b) $x' = x^2 - x + y, \quad y' = -yx^2 - y.$

Topic 30. Systems: population models

30.1. The main tourist attraction at Monet Gardens is Pristine Acres, an expanse covered with artfully arranged wildflowers. Unfortunately, the flower stems are the favorite food of the Kandinsky borer; the flower and borer populations fluctuate cyclically in accordance with Volterra's predator-prey equations. To boost the wildflower level for the tourists, the director wants to fertilize the Acres, so that the wildflower growth will outrun that of the borers. Assume that fertilizing would boost the wildflower growth rate (in the absence of borers) by 25 percent. What do you think of this proposal? Using suitable units, let x be the wildflower population and y be the borer population. Take the equations to be

$$\begin{aligned} x' &= ax - pxy \\ y' &= -by + qxy, \end{aligned}$$

where a, b, p, q are positive constants.

30.2. Let $x(t)$ be the population of sharks of the coast of Massachusetts and $y(t)$ the population of fish. Assume that the populations satisfy the Volterra predator-prey equations

$$x' = ax - pxy; \qquad y' = -by + qxy, \qquad \text{where } a,\ b,\ p,\ q, \text{ are positive.}$$

Assume time is in years and a and b have units 1/years.

Suppose that warming waters kill the 10% of both the fish and the sharks each year. Show that the shark population actually increases.

30.3. Consider the system of equations

$$x'(t) = 39x - 3x^2 - 3xy; \qquad y'(t) = 28y - y^2 - 4xy.$$

The four critical points of this system are (0,0), (13,0), (0,28), (5,8).

(a) Show that the linearized system at (0,0) has eigenvalues 39 and 28. What type of critical point is (0,0)?

(b) Linearize the system at (13,0); find the eigenvalues; give the type of the critical point.

(c) Repeat Part (b) for the critical point (0,28).

(d) Repeat Part (b) for the critical point (5,8).

30.4. The equations for this system are

$$\begin{aligned} x' &= x^2 - 2x - xy \\ y' &= y^2 - 4y + xy \end{aligned}$$

(a) If this models two populations, what would happen to each of the populations in the absence of the other?

(b) There are four critical points. Find and classify them

(c) Sketch a phase portrait of the system.

30.5. The system for this equation is

$$x' = 4x - x^2 - xy$$
$$y' = -y + xy$$

(a) This is a predator-prey system. Which of x and y represents the predator population?

(b) What would happen to the predator population in the absence of prey? What about the prey population in the absence of predators?

(c) There are three critical points. Find and classify them

(d) Sketch a phase portrait of this system.

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