# ES.1803 Practice Solutions – Quiz 1, Spring 2024

**Problem 1.**
*Solve the DE* $\dfrac{dy}{dx} = xy^2$.

**Solution:** Separable: $\dfrac{dy}{y^2} = x dx$. So, $-\dfrac{1}{y} = x^2/2 + C$. Solving for $y$ $\boxed{y(x) = -\dfrac{2}{x^2 + 2C}}$.

There are also lost solutions when $y^2 = 0$, i.e., $\boxed{y(x) = 0.}$

**Problem 2.**
*Solve the following DE.* $\cos(x)\dfrac{dy}{dx} + \sin(x)\cos(y) = 0$

**Solution:** Separation of variables: $\dfrac{dy}{\cos(y)} = \dfrac{-\sin(x)dx}{\cos(x)}$.

Integrating: $\ln(\sec(y) + \tan(y)) = \ln|\cos(x)| + C$.

It's hard to solve this for $y$ as a function of $x$. Here's one answer: $\boxed{\sec(y) + \tan(y) = C_1|\cos(x)|}$.

There are lost solutions when $\cos(y) = 0$, i.e., the lost solutions are $\boxed{y(x) = \pi/2,\ y(x) = 3\pi/2\ \ldots}$

**Problem 3.**
*A species $S$ in a constrained envrironment cannot grow exponentially forever. Call the population over time $x(t)$. Assume the growth rate is not constant, but depends on $x$. Specifically,*

$$\textit{growth rate} = aM - ax.$$

**(a)** *What are the units on $M$ and $a$*

**Solution:** A growth rate has units of 1/time. So if $ax$ has units of 1/time, then $\boxed{a \text{ has units } 1/(\text{time} \cdot S).}$

Likewise, if $aM$ has units of 1/time, then $\boxed{M \text{ has units of } S}$

**(b)** *Model the population with a DE.*

**Solution:** Growth rate measures the instantaneous rate the population is increasing as a fraction of the population. So, if the growthrate is $k$, then

$$x' = kx$$

In this case, we are given that $k = (aM - ax)$. This is not constant, but that doesn't change the DE. We get,

$$x' = (aM - ax)x$$

This is called a *logistic* population model.

**Problem 4.**
*A salt solution of strength 2 grams per liter is flowing into a 50 liter tank, and the solution in the tank is being pumped out, both at the (same) time-varying rate of* $\dfrac{1}{1+t}$ *liters per minute.*

**(a)** *With the usual instantaneous-mixing assumptions, derive the DE for the rate of change of the amount of salt $x = x(t)$ (in grams) in the tank with respect to time (in minutes).*

**Solution:** We'll show the DE is $\boxed{\frac{dx}{dt} + \left(\frac{1}{50}\right)\left(\frac{1}{1+t}\right) x = \frac{2}{1+t}}$.

Net rate of change of salt in tank $= x' =$ (salt rate in) − (salt rate out).

Salt rate in = (flow rate in) · (concentration in) $= \frac{1}{1+t} \cdot 2$.

Salt rate out = (flow rate out) · (concentration out) $= \frac{1}{1+t} \cdot \frac{x}{V}$, where $V = 50$ is the volume of solution in the tank.

This gives the equation: $\frac{dx}{dt} = \frac{2}{1+t} - \frac{1}{50}\frac{1}{1+t} x$ or $\boxed{\frac{dx}{dt} + \frac{1}{50}\frac{1}{1+t} x = \frac{2}{1+t}}$.

**(b)** *Solve this DE explicitly for $x = x(t)$ with the IC $x(0) = 0$ (i.e., starting off with pure water in the tank).*

**Solution:** $x' + \frac{1}{50}\frac{1}{1+t} x = \frac{2}{1+t}$. is a linear first-order DE in standard form.

We use the variation of parameters formula:

Homogeneous solution: $x_h(t) = e^{-\int p(t)dt} = e^{-\int \frac{1}{50}\frac{1}{1+t}dt} = e^{-\frac{1}{50}\ln(1+t)} = (1+t)^{-1/50}$.

$$\begin{aligned} x(t) &= (1+t)^{-1/50} \int (1+t)^{\frac{1}{50}} 2/(1+t)dt \\ &= (1+t)^{-1/50} \int 2(1+t)^{\frac{1}{50}-1} \\ &= (1+t)^{-1/50}(100(1+t)^{1/50} + C) \\ &= 100 + C(1+t)^{-1/50}. \end{aligned}$$

The initial condition, $x(0) = 0$. So, $C = -100$. and $\boxed{x(t) = 100(1 - (1+t)^{-1/50})}$.

**(c)** *Using both (i) the solution $x = x(t)$ found in Part (b) and (ii) an argument from 'first principles' (i.e., directly from the physical situation described here), answer the following question: What happens to the amount of salt in the tank in the long-run over time? In particular, does it approach a final limiting value? If so, what is this value?*

*(The idea is to do both (i) and (ii) and show they give the same prediction for the long-term behavior.)*

**Solution:** (i) Looking at the formula for $x(t)$ we see that $x(t) \to 100$ (grams) as $t \to \infty$.

(ii) The concentration of the in-flow is 2 gms/liter. Over time, the concentration in the tank will approach this value. Since the tank is 50 liters in volume, the amount of salt in the tank will approach 2 gms/liter · 50 liters = 100 gms.

**Problem 5.**
*Assume $y_1$ is a solution to the DE $y' + e^t y = 0$. Also assume that $y_p$ is a solution to the DE $y' + e^t y = t^4$. Prove that all the functions $y = y_p + cy_1$, where $c$ is any constant, are also solutions to $y' + e^t y = t^4$.*

**Solution:** Substitute $y = y_p + cy_1$ into the DE:

$$(y_p + cy_1)' + e^t(y_p + cy_1) = (y_p' + e^t y_p) + (cy_1' + ce^t y_1) = t^4 + 0 = t^4 \qquad \blacksquare.$$

**Problem 6.**
*Suppose that the functions* $y_1 = t$ *and* $y_2 = \dfrac{1}{t}$ *both satisfy a certain inhomogeneous first-order linear DE. Write down the general solution to the DE.*

**Solution:** We know that the general solution to an inhomogeneous first-order linear DE is of the form $y_p(t) + cy_h(t)$, where $y_p$ is one solution to the inhomogeneous DE and $y_h$ is a solution to the associated homogeneous DE.

By linearity $y_h = y_1 - y_2$ satisfies the related homogeneous equation. Thus, $\boxed{y = y_1 + cy_h = t + c(t - 1/t)}$ is the general solution.

*End of practice quiz solutions*