ES.1803 Practice Questions – Quiz 2, Spring 2024
Covers Topics 4-8

This will probably take far longer than 1 hour; the actual quiz will be considerably shorter. Not every possible question is covered. For more questions, you can also look at the in-class problems, problem section worksheets and psets.

Be wise in carrying out computations. If you get a problem to a point where you are sure –*emphasize sure*– you can carry out the computation, then stop and go on to the next problem. At the end you can check the solutions to see that you had correctly set up the computation.

Problem 1.

(a) Compute the following *real* function of x: $\operatorname{Im}\left(\frac{e^{(3+2i)x}}{3+2i}\right)$.

(As usual, $\operatorname{Im}(z)$ denotes the imaginary part of the complex number z.)

(b) Use the result of Part (a) to compute the integral $\int e^{3x}\sin(2x)\,dx$ using the complex exponential.

Problem 2. Find the 3 cube roots of 1 by locating them on the unit circle and using basic trigonometry.

Problem 3. Use Euler's formula to derive the trig addition formulas for sin and cos.

Problem 4.

(a) Find the general *real-valued* solution to the DE $y'' + 4y' + 13y = 0$. Also find the solution satisfying the intial conditions (IC) $y(0) = 1$, $y'(0) = 0$.

(b) For what values of b will all the (non-zero) solutions to $y'' + by' + 13y = 0$ display oscillatory behavior?

(c) For these oscillatory solutions, in *theory* how many times does each solution cross the positive t-axis. If this DE is modeling some real-world situation, what actually happens to the quantity $y = y(t)$ in the long run?

Problem 5.

(a) Give the general *real-valued* solution to the DE $x'' + 7x' + 12x = 2e^t$.

(b) (5) Give the general *real-valued* solution to the DE $x'' + 7x' + 12x = 2e^t + 3e^{-t}$.

Problem 6. Consider the IVP $x'' + bx' + 4x = 0$; $x(0) = 1$; $x'(0) = 0$.

Sketch a graph of the solution in the following cases. (You don't need to solve the DE completely or give a detailed sketch. You do need to give a small amount of explanation.)

(a) (5) When $b = 1$

(b) (5) When $b = 6$.

Problem 7. Given the DE $y'' + 4y' + 5y = 8\cos(2t)$:

(a) Find the general solution to the DE.

(b) What is the periodic solution? Give your answer in amplitude-phase form.

(c) Show that, no matter what the initial conditions, the system always settles down to the periodic solution.

Problem 8. Find the general solution to $x' + 3x = t^2 + 3$

Problem 9. Find one solution to $x''' + 3x'' + 2x' + 5x = 4$.

Problem 10. Let $P(D) = D^2 + bD + I$ where $D = \dfrac{d}{dt}$ and $b > 0$.

(a) For what range of the values of b will the solutions to $P(D)y = 0$ exhibit oscillatory behavior?

(b) Describe the different types of graphs, $y = y(t)$, one gets for values of $b > 0$.

(c) For $b = 1$, solve the DE $P(D)\,y = f(t)$ for a particular solution $y_p(t)$ where f is the following:

(i) $f(t) = 2\,e^{-t}\sin(2t)$ (ii) $f(t) = 2\,e^{-t}\cos(2t)$ (iii) $f(t) = t + 3$.

(d) For $b = 2$, find the general solution of the DE $P(D)\,y = 0$

Problem 11. Consider the DE $y'' + 2y' + ky = 0$.

(a) For which values of k will there be solutions $y(t)$ with infinitely many zeros?

(b) For such solutions, express in terms of k the t-distance between successive zeros.

(c) For which values of k will $\lim_{t\to\infty} y(t) = 0$ for all solutions $y(t)$? (Indicate reason.)

(d) For which values of k will $\lim_{t\to\infty} y(t) = 0$ for at least one nontrivial solution $y(t)$? Why?

Problem 12. Assume L is a linear differential operator and y_1 is a solution to the DE $Ly = 0$. Prove that if y_p is a solution to the DE $Ly = f$, then so are all the functions $y_p + cy_1$, where c is any constant.

Problem 13. (*Parts (a) and (b) are not related*)

(a) Suppose that the functions $y_1 = t$ and $y_2 = \dfrac{1}{t}$ both satisfy a certain inhomogeneous first-order linear DE. Write down the general solution to the DE.

(b) Let T be the operator defined by $Tf = f^2$.

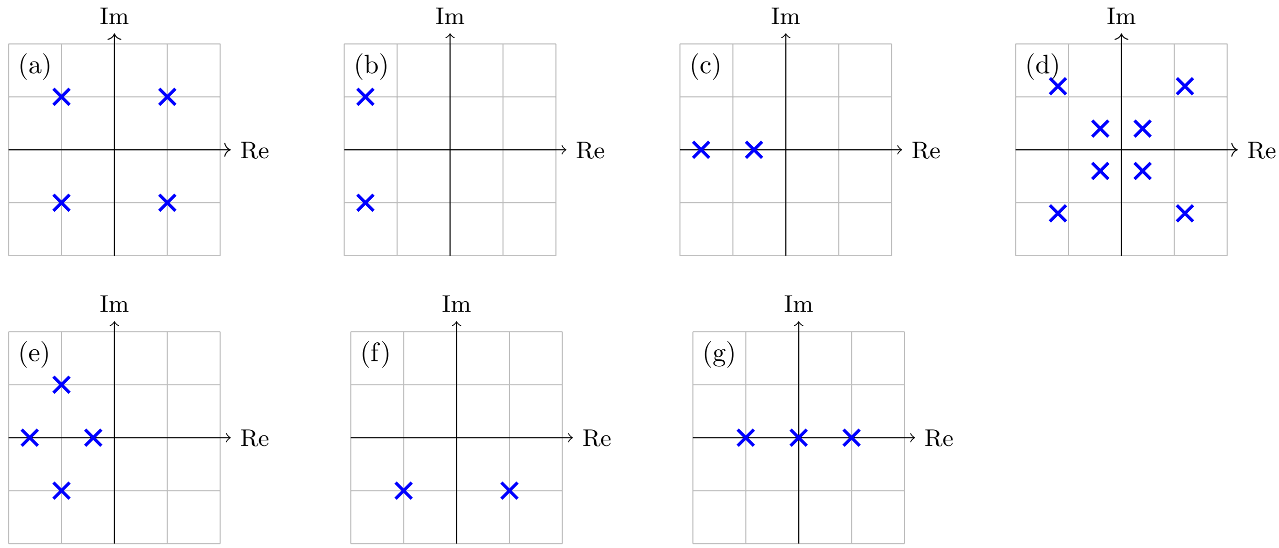
(i) Show that the operators T and D do not commute.

(ii) Is T a linear operator? (You must give an explanation.)

Problem 14.

(a) In this problem we consider the linear constant coefficient DE $P(D)x = 0$.

Assume $P(D)$ is of arbitrary order. Each of the following plots are in the complex plane and the crosses give the locations of the zeros of $P(r)$. If the plot comes from a stable system label it as 'stable'. If not, label it as 'unstable'.



(b) Assume all the pole diagrams are on the same scale, which of the stable systems decays to equilibrium the fastest.

Problem 15. Water is being heated at a rate of 5°C per minute. It is simultaneously cooling at a rate proportional to the difference between its current temperature $T(t)$ and the ambient temperature A in which it is sitting.

(a) (5) Write down the DE for the temperature $T(t)$. Define all letters used, with units.

(b) (5) What is the long-range behavior of $T(t)$ as t increases, and why?

End of practice quiz