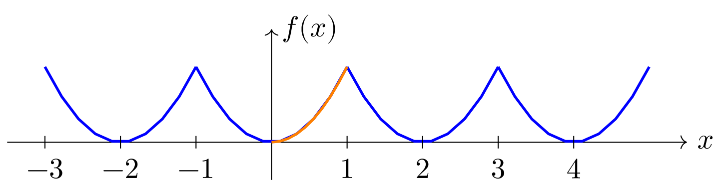
ES.1803 Problem Section 11, Spring 2024 Solutions

Integral table

$$\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}$$
$$\int t\sin(\omega t)\,dt = -\frac{t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}$$
$$\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}$$
$$\int t^2\sin(\omega t)\,dt = -\frac{t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}$$
$$\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$
$$\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$$
$$\int \cos(at)\sin(bt)\,dt = -\frac{1}{2}\left[\frac{\cos((a+b)t)}{a+b} - \frac{\cos((a-b)t)}{a-b}\right]$$
$$\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$$
$$\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$$
$$\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$$

Problem 23.1. *Find the Fourier cosine series for the function $f(x) = x^2$ on $[0, 1]$. Graph the function and its even period 2 extension.*

Solution: The graph of the even, period 2 extension is shown below. $f(x)$ is shown as the orange segment above the interval $[0, 1]$.



We have $L = 1$. The cosine coefficients are computed as usual. (Or just use a table of integrals.)

$$\begin{aligned} a_0 &= 2\int_0^1 f(x)\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}. \\ a_n &= 2\int_0^1 f(x)\cos(n\pi x)\,dx = 2\int_0^1 x^2\cos(n\pi x)\,dx \\ &= 2\left[\frac{x^2\sin(n\pi x)}{n\pi} + \frac{2x\cos(n\pi x)}{(n\pi)^2} - \frac{2\sin(n\pi x)}{(n\pi)^3}\right]_0^1 \\ &= \frac{4(-1)^n}{(n\pi)^2}. \end{aligned}$$

Thus, $f(x) = \dfrac{1}{3} + 4\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\pi^2 n^2}\cos(n\pi x)$.

Problem 23.2. *Find Fourier cosine series for* $\sin(x)$ *on* $[0,\pi]$.

Cosine series. $L = \pi$, Using the formula for a_n:

$a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin(x)\,dx = \left[-\frac{2}{\pi}\cos(x)\right|_0^{\pi} = \frac{4}{\pi}.$

By using an integral table $\left(\text{or applying the formula: } \cos(ax)\sin(bx) = \dfrac{\sin((a+b)x) - \sin((a-b)x)}{2}\right)$ with $a = n$ and $b = 1$, we get:

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\,\cos(nx)\,dx = -\frac{1}{\pi}\left[\frac{\cos((n+1)x)}{n+1} - \frac{\cos((n-1)x)}{n-1}\right|_0^{\pi} = \begin{cases} 0 & \text{for odd } n > 0 \\ \frac{-4}{\pi(n^2-1)} & \text{for even } n > 0. \end{cases}$$

(You have to be careful with $n = 1$, but the formula is correct.)

Thus,

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos(2x)}{3} + \frac{\cos(4x)}{15} + \frac{\cos(6x)}{35} + \ldots\right) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n>0,\text{ even}} \frac{\cos(nx)}{n^2-1}.$$

Important. This is only valid where $f(x)$ is defined, i.e., on $[0,\pi]$.

Problem 24.3. **(a)** *Solve* $x'' + 2x' + 9x = g(t)$*, where* $g(t)$ *is the period 2 triangle wave with* $g(t) = |t|$ *on* $[-1,1]$*. Find the Fourier series of* g *by using* $g(t) = f(\pi t)/\pi$*, where* f *is the standard period* 2π *triangle wave* $f(t) = |t|$ *on* $[-\pi,\pi]$.

Solution: We know $f(t) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\displaystyle\sum_{n\text{ odd}} \frac{\cos(nt)}{n^2}$. So,

$$g(t) = \frac{f(\pi t)}{\pi} = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n\text{ odd}} \frac{\cos(n\pi t)}{n^2}.$$

(Or you can just compute the integrals for the coefficients.)

Use the SRF to solve for each piece: (For ease of writing, we'll leave out the coefficients here and reintroduce them in the superposition step.)

$$x_n'' + 2x_n' + 9x_n = \cos(n\pi t).$$

First we find $P(in)$ in polar form: $P(i\pi n) = 9-(\pi n)^2+2i\pi n = \sqrt{(9-\pi^2n^2)^2+4\pi^2n^2}\, e^{i\phi(n)}$, where $\boxed{\phi(n) = \text{Arg}(P(in)) = \tan^{-1}(2n\pi/(9-\pi^2n^2)) \text{ in Q1 or Q2.}}$

So, $x_{n,p}(t) = \dfrac{\cos(n\pi t - \phi(n))}{\sqrt{(9-\pi^2n^2)^2+4\pi^2n^2}}$.

Separate calculation for $n=0$: $x_0'' + 2x_0' + 9x_0 = \dfrac{1}{2} \Rightarrow x_{0,p}(t) = 1/18$.

Superposition:

$$x_p(t) = x_{0,p} - \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{x_{n,p}}{n^2} = \frac{1}{18} - \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{\cos(n\pi t - \phi(n))}{n^2\sqrt{(9-\pi^2n^2)^2+4\pi^2n^2}}.$$

(Don't forget you need to include n in $\phi(n)$.)

(b) *Is there a term in the Fourier series for g whose frequency is near the natural frequency of the system modeled by the DE? For the response found in Part (a), does this term have the largest amplitude?*

Solution: The answers are yes and yes. The undamped, unforced system is $x'' + 9x = 0$. This has natural frequency $\omega_0 = 3$. The $n=1$ term in the Fourier series for $g(t)$ has frequency $\pi \approx 3.14$ which is close to ω_0.

The amplitude of the response to the nth term is $\dfrac{4}{\pi^2n^2\sqrt{(9-\pi^2n^2)^2+4\pi^2n^2}}$. It is clear that the denominator is smallest for $n=1$, therefore the amplitude is largest.

Problem 23.4. *Find the Fourier sine series for $f(x) = 1$ on $[0,\pi]$.*

Solution: The odd extension is the square wave $\Rightarrow f(x) = \dfrac{4}{\pi}\displaystyle\sum_{n\, odd} \frac{\sin(nx)}{n}$.

Problem 24.5. *Solve $x' + kx = f(t)$, where $f(t)$ is the period 2π triangle wave with $f(t) = |t|$ on $[-\pi,\pi]$. (You can use the known series for $f(t)$.)*

Solution: We know the Fourier series for $f(t)$, but we'll sketch the computation.

$f(t)$ is even, so $b_n = 0$. We use the evenness to simplify the integral for the cosine coefficients

$$a_0 = \frac{2}{\pi}\int_0^\pi t\,dt = \pi, \qquad a_n = \frac{2}{\pi}\int_0^\pi t\cos(nt)\,dt = \begin{cases} -\frac{4}{\pi n^2} & \text{if } n \text{ odd} \\ 0 & \text{if } n \neq 0 \text{ even} \end{cases}$$

So the DE is: $x' + kx = \dfrac{\pi}{2} - \dfrac{4}{\pi}\displaystyle\sum_{n\, odd} \frac{\cos(nt)}{n^2}$.

Superposition: We'll solve for each piece first: $x_n' + kx_n = \dfrac{4}{n^2\pi}\cos(nt)$

We use the sinusoidal response formula (SRF). First compute $P(in)$ in polar form.

$$P(in) = k + in = \sqrt{k^2+n^2}\, e^{i\phi(n)}, \text{ where } \boxed{\phi(n) = \text{Arg}(P(in)) = \tan^{-1}(n/k) \text{ in Q1}}.$$

The SRF gives: $\boxed{x_{n,p}(t) = \dfrac{4\cos(nt-\phi(n))}{\pi n^2 |P(in)|} = \dfrac{4\cos(nt-\phi(n))}{\pi n^2 \sqrt{k^2+n^2}}}$.

Separate calculation for $n = 0$: $x_0' + kx_0 = \pi/2 \quad \Rightarrow x_{0,p}(t) = \pi/2k$.

Superposition:

$$\boxed{x_p(t) = x_{0,p} - \sum_{n \text{ odd}} x_{n,p} = \frac{\pi}{2k} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(nt-\phi(n))}{n^2\sqrt{k^2+n^2}}}.$$

Extra problems if time.

Problem 24.6. *Solve* $x'' + 4x = \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}$*. Look out for resonance.*

Solution: Solve this in pieces: $x_n'' + 4x_n = \cos(nt)$: (For practice, we leave out the coefficient $1/n^2$. We'll need to include it in the superposition at the end.)

We'll need $P(in)$ in polar form.

$$P(in) = 4 - n^2 = |4-n^2|e^{i\phi(n)}, \text{ where } \phi(n) = \text{Arg}(P(in)) = \begin{cases} 0 & \text{if } n = 1 \\ \pi & \text{if } n \geq 3 \\ \text{undefined} & \text{if } n = 2 \end{cases}$$

Using the SRF, for $n \neq 2$, we have $x_{n,p}(t) = \dfrac{\cos(nt-\phi(n))}{|P(in)|} = \dfrac{\cos(nt-\phi(n))}{|4-n^2|}$.

For $n = 2$, we need to use the extended SRF:

$P'(r) = 2r$. So, $P'(2i) = 4i = 4e^{i\pi/2}$. Now the extended SRF gives $x_{2,p}(t) = \dfrac{t\cos(2t-\pi/2)}{4}$.

Summarizing, we have $x_{n,p}(t) = \begin{cases} \frac{\cos(t)}{3} & \text{for } n = 1 \\ \frac{\cos(2t-\pi/2)}{4} & \text{for } n = 2 \\ \frac{\cos(nt-\pi)}{|4-n^2|} & \text{for } n \geq 3. \end{cases}$

Now, by superposition,

$$x_p(t) = \sum_{n=1}^{\infty} \frac{x_{n,p}(t)}{n^2} = \frac{\cos(t)}{3} + \frac{t\cos(2t-\pi/2)}{16} + \sum_{n=3}^{\infty} \frac{\cos(nt-\pi)}{n^2|4-n^2|}.$$

Finally, using $\cos(2t-\pi/2) = \sin(2t)$ and $\cos(nt-\pi) = -\cos(nt)$, we can simplify the expression for $x_p(t)$:

$$x_p(t) = \frac{\cos(t)}{3} + \frac{t\sin(2t)}{16} - \sum_{n=3}^{\infty} \frac{\cos(nt)}{n^2|4-n^2|}$$

Problem 23.7. *Find the Fourier series for the standard square wave shifted to the left so it's an even function, i.e., $sq(t + \pi/2)$.*

Solution: Call the standard period 2π, odd, amplitude 1 square wave $sq(t)$. We know that $sq(t) = \frac{4}{\pi}\sum_{n\,odd} \frac{\sin(nt)}{n}$

Our function is $f(t) = sq(t+\pi/2) = \frac{4}{\pi}\sum_{n\,odd}\frac{\sin(n(t+\pi/2))}{n} = \frac{4}{\pi}\left(\cos(t) - \frac{\cos(3t)}{3} + \frac{\cos(5t)}{5} - \ldots\right)$.

This last equation follows because

$$\sin(\theta + \pi/2) = \cos(\theta), \quad \sin(\theta + 3\pi/2) = -\cos(\theta), \quad \sin(\theta + 5\pi/2) = \cos(\theta) \ldots.$$

(You can see this either using the trig identity for $\sin(a + b)$ or by thinking about shifting a sine curve to the left by an odd multiple of $\pi/2$.)

Problem 22.8. **(a)** *Compute the Fourier series for the even, period 2π function, with $f(t) = \pi t - t^2$ on $[0, \pi]$. The integral table provided should help.*

Solution: Since $f(t)$ is even, $b_n = 0$.

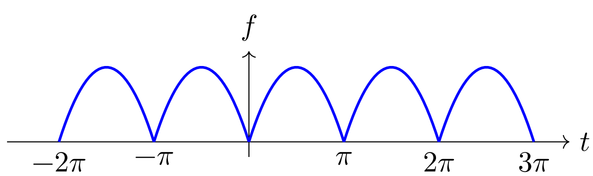
Using the integral table to compute the integrals, we find

$$a_n = \begin{cases} \frac{2}{\pi}\int_0^{\pi}(\pi t - t^2)\cos(nt)\,dt = -4/n^2 & \text{for } n \text{ even, } n \neq 0 \\ \frac{2}{\pi}\int_0^{\pi}(\pi t - t^2)\cos(nt)\,dt = 0 & \text{for } n \text{ odd} \\ \frac{2}{\pi}\int_0^{\pi}\pi t - t^2\,dt = \pi^2/3 & \text{for } n = 0. \end{cases}$$

So, $f(t) = \frac{\pi^2}{6} - 4\sum_{n \text{ even}}\frac{\cos(nt)}{n^2}$.

(b) *Carefully sketch the graph of the Fourier series.*

The function $f(t)$ is continuous at all t, so the Fourier series converges to $f(t)$



(c) *Challenge: Can you explain why the odd cosine coefficients are 0?*

Solution: This is really a period π function so its Fourier series has fundamental angular frequency 2.

Problem 22.9. *The function $f(t)$ has period π. Over the interval $0 \le x < \pi$ we have $f(t) = \sin(t)$. Sketch the graph of $f(t)$ over 3 full periods and find the Fourier series for $f(t)$*

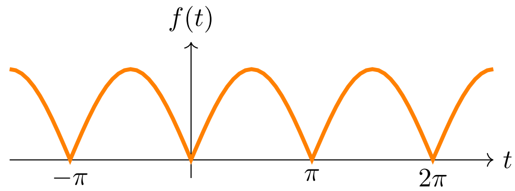
Solution: This is an even function, so we only need to compute the cosine coefficients (a_n). We don't show all the details of the integrations. An integral table will help here

We have the half-period $L = \pi/2$. In this case, I think it is easiest to integrate over a full period $[0, \pi]$ rather than use the doubling trick for even functions.

$$a_0 = \frac{1}{\pi/2}\int_0^{\pi} \sin(t)\,dt = -\frac{2}{\pi}\left[\cos(t)\right]_0^{\pi} = \frac{4}{\pi}$$

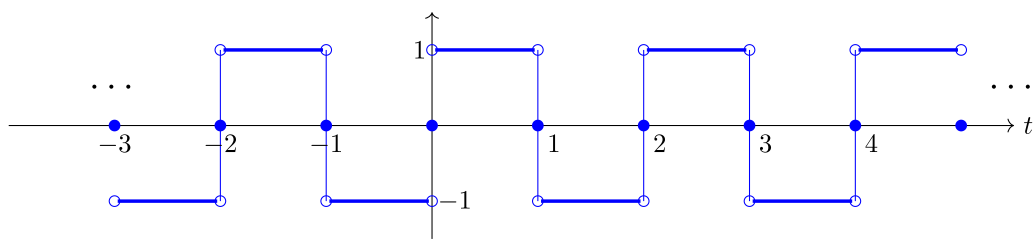
$$a_n = \frac{1}{\pi/2}\int_0^{\pi} \sin(t)\cos(2nt)\,dt = \frac{2}{\pi}\cdot\frac{1}{2}\left(-\frac{\cos((2n+1)t)}{2n+1} + \frac{\cos((2n-1)t)}{2n-1}\right]_0^{\pi} = -\frac{4}{\pi(4n^2-1)}$$

So, $f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2nt)}{4n^2-1}$



Problem 22.10. *Let $f(t)$ be the odd, period 2, amplitude 1 square wave. Carefully sketch the graph of the Fourier series.*

Solution: The key to the sketch is to put dots at the midpoint of each jump and open circles at the ends of each line segment.



Problem 22.11. *Recall the Fourier series for the period 2π triangle wave tri(t), where tri(t) $= |t|$ for $-\pi \le t \le \pi$:*

$$\text{tri}(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}}\frac{\cos(nt)}{n^2}.$$

Set $t = 0$ and show $\sum_{n \text{ odd}}\frac{1}{n^2} = \frac{\pi^2}{8}$. (This is only for fun, we will not test on this sort of problem.)

Solution: We know tri(0) = 0. Putting $t = 0$ in the Fourier series gives

$$\text{tri}(0) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}}\frac{1}{n^2} = 0.$$

A small amount of algebra now shows that $\sum_{n \text{ odd}}\frac{1}{n^2} = \frac{\pi^2}{8}$..