ES.1803 Problem Section 2, Spring 2024

Problem 3.1. Solve the DE $x'+2x = f(t)$, $x(0) = 0$, where $f(t) = \begin{cases} 6 & \text{for } 0 \le t < 1 \\ 0 & \text{for } 1 \le t < 2 \\ 6 & \text{for } 2 \le t. \end{cases}$

Problem 4.2. (Polar coordinates)
Write $z = -1 + \sqrt{3}i$ in polar form.

Problem 4.3. (Trig triangle)
Draw and label the triangle relating rectangular with polar coordinates.

Problem 4.4. (Roots)
Find all fifth roots of -2. Give them in polar form. Draw a figure showing the roots in the complex plane.

Problem 4.5. (Complex replacement or complexification)
Compute $I = \int e^{4x} \cos(3x)\, dx$ using complex techniques.

Problem 4.6. Using the polar form, explain why $|z^n| = |z|^n$ and $\arg(z^n) = n \arg(z)$ for n a positive integer.

Problem 4.7. **(a)** Write $\cos(\pi t) - \sqrt{3}\sin(\pi t)$ in the form $A\cos(\omega t - \phi)$.

(b) Write $5\cos\left(3t + \frac{3\pi}{4}\right)$ in the form $a\cos(\omega t) + b\sin(\omega t)$.

(In each case, begin by drawing a right triangle with sides a and b, angle ϕ, hypotenuse A.)

Extra problems if time.

Problem 4.8. (Polar coordinates)
We know $-1 + \sqrt{3}\,i = 2e^{i2\pi/3}$. Use this to answer the following questions.

(a) Compute the product $(-1 + \sqrt{3}i)(a + bi)$ (where a, b are real).

Describe geometrically what multiplying by $-1 + \sqrt{3}i$ does.

(b) What are the polar coordinates of $(-1+\sqrt{3}i)(a+bi)$ in terms of the polar coordinates of $a + bi = re^{i\theta}$?

(c) Describe the sequence of powers of $-1 + \sqrt{3}i$, positive and negative.

Problem 4.9. Compute $\dfrac{1}{-2+3i}$ in polar form. Convert the denominator to polar form first. Be sure to describe the polar angle precisely.

Problem 4.10. Make up and solve some simple algebra problems involving addition, subtraction, division, magnitude, complex conjugation.

Problem 4.11. Write $3e^{i\pi/6}$ in rectangular coordinates.

Problem 4.12. Find a formula for $\cos(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

Problem 4.13. The point of this problem is to help you distinguish between taking the real part of a function and finding which members of a *family of functions* are real-valued.

(a) Show the inverse Euler formulas are true:

$$\cos(t) = (e^{it} + e^{-it})/2, \qquad \sin(t) = (e^{it} - e^{-it})/2i.$$

(b) Find all the real-valued functions of the form $\tilde{c}_1 e^{it} + \tilde{c}_2 e^{-it}$, where $\tilde{c}_1$ and $\tilde{c}_2$ are complex constants.

Problem 4.14. Find all the real-valued functions of the form $x = \tilde{c}e^{(2+3i)t}$.

Problem 4.15. Find the 3 cube roots of 1 by locating them on the unit circle and using basic trigonometry.

Problem 4.16. Express in the form $a + bi$ the 6 sixth roots of 1.

Problem 4.17. Use Euler's formula to derive the trig addition formulas for sin and cos.

Problem 4.18. Suppose $z^n = 1$. What must $|z|$ be? What are the possible values of $\arg(z)$, if $z^n = 1$?

Problem 4.19. Find the cube roots of i.

Problem 4.20. By using $(e^{it})^4 = e^{4it}$ and Euler's formula, find an expression for $\sin(4t)$ in terms of powers of $\cos(t)$ and $\sin(t)$,

Problem 4.21. Trajectories of $e^{(a+bi)t}$ can vary a lot, depending upon the value of the complex number $a + bi$. The "Complex Exponential" Mathlet shows this clearly. Invoke this applet if you can: https://mathlets.org/mathlets/complex-exponential/. You can use it to gain insight into the following questions.

(a) Sketch the trajectory of the complex-valued function $e^{(-1+2\pi i)t}$, and the graphs of its real and imaginary parts.

(b) For each of the following shapes, decide on all the values of $a + bi$ for which the trajectory of $e^{(a+bi)t}$ has this shape.

(i) A circle centered at 0, traversed counterclockwise. What circles are possible?

(ii) A circle centered at 0, traversed clockwise.

(iii) A ray (straight half line) heading away from the origin.

(iv) A curve heading to zero as $t \to \infty$.

Problem 4.22. Write $\cos(2t) + \sin(2t)$ in the form $A\cos(\omega t - \phi)$.