ES.1803 Problem Section 3, Spring 2024 Solutions

Problem 5.1. **(a)** *Solve $x'' - 8x' + 7x = 0$ using the characteristic equation method.*

Solution: (Model solution) Characteristic equation: $r^2 - 8r + 7 = 0$.

Roots: $r = 7,\ 1$.

General real-valued solution: $x(t) = c_1 e^{7t} + c_2 e^{t}$.

(b) *Solve $x'' + 2x' + 5x = 0$ using the characteristic equation method.*

Solution: Characteristic equation: $r^2 + 2r + 5 = 0$.

Roots: $r = (-2 \pm \sqrt{4 - 20})/2 = -1 \pm 2i$.

General real-valued solution: $x(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$.

(c) *Assume the polynomial $r^5 + a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0$ has roots*

$$0.5, \quad 1, \quad 1, \quad 2 \pm 3i.$$

Give the general real-valued solution to the homogeneous constant coefficient DE

$$x^{(5)} + a_4 x^{(4)} + a_3 x^{(3)} + a_2 x'' + a_1 x' + a_0 x = 0.$$

Solution: Since we are given the roots, we can write the general solution directly:

$$x(t) = c_1 e^{0.5t} + c_2 e^{t} + c_3 t e^{t} + c_4 e^{2t}\cos(3t) + c_5 e^{2t}\sin(3t).$$

Problem 5.2. **(Unforced second-order physical systems)**
The DE $x'' + bx' + 4x = 0$ models a damped harmonic oscillator. For each of the values $b = 0, 1, 4, 5$ say whether the system is undamped, underdamped, critically damped or overdamped.

Sketch a graph of the response of each system with initial condition $x(0) = 1$ and $x'(0) = 0$. (It is not necessary to find exact solutions to do the sketch.)

Say whether each system is oscillatory or non-oscillatory.

Solution: The characteristic roots are $\dfrac{-b \pm \sqrt{b^2 - 16}}{2}$. We call the term under the square root the discriminant.

$b = 0$: The system is undamped and oscillatory (in fact sinusoidal).

$b = 1$: The discriminant $= 1 - 16 < 0$, so the roots are complex, which implies the system is underdamped and oscillatory.

$b = 5$: The discriminant is positive, so the roots are real, which implies system overdamped and non-oscillatory.

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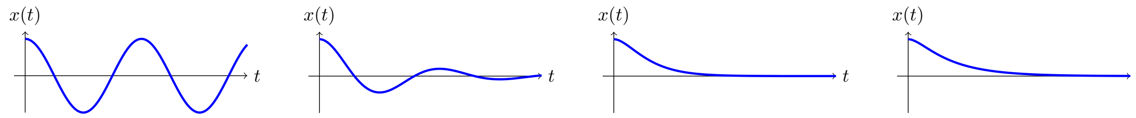
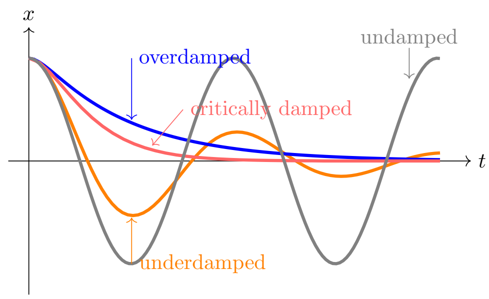
Here are plots of each of these solutions starting from $x(0) = 1$, $x'(0) = 0$.

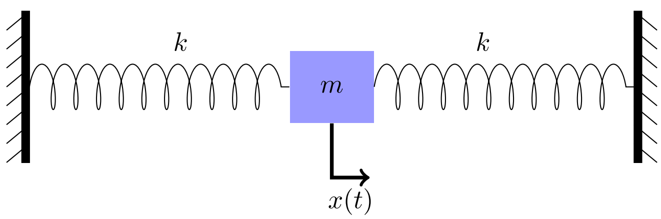


$b = 0$: undamped $b = 1$: underdamped $b = 4$: crit. damped $b = 5$: overdamped

The following figure is from the Topic 5 notes. It shows the different types of damping, though not necessarily using the coefficients in this problem. Note, that the initial conditions are all the same and, the initial velocity $x'(0) = 0$ causes all the graphs to have a horizontal tangent at $t = 0$.



Problem 5.3. *In the spring system below, both springs are unstretched when the position of the mass is $x = 0$, which is exactly in the middle. Write down a DE modeling the position of the mass over time.*



Solution: When the mass is at $x > 0$ then the left spring is stretched by x and the right spring is compressed by x, so the force on the mass is $m\ddot{x} = -2kx$ or $m\ddot{x} + 2kx = 0$.

Problem 5.4. *State and verify the superposition principle for $mx'' + bx' + kx = 0$, (m, b, k constants).*

Solution: Superposition principle for linear, homogeneous DEs:

If x_1 and x_2 are solutions to the DE then so are all linear combinations $x = c_1x_1 + c_2x_2$.

Proof. Plug x into the DE and then chug through the algebra to show that x is a solution.

$$\begin{aligned}
mx'' + bx' + kx &= m(c_1x_1 + c_2x_2)'' + b(c_1x_1 + c_2x_2)' + k(c_1x_1 + c_2x_2) \\
&= c_1mx_1'' + c_2mx_2'' + c_1bx_1' + c_2bx_2' + c_1kx_1 + c_2kx_2 \\
&= c_1\underbrace{mx_1'' + bx_1' + kx_1}_{\substack{\text{0 by assumption that} \\ x_1 \text{ is a solution}}} + c_2\underbrace{mx_2'' + bx_2' + kx_2}_{\substack{\text{0 by assumption that} \\ x_2 \text{ is a solution}}} \\
&= 0 \qquad \blacksquare
\end{aligned}$$

Problem 5.5. *A constant coefficient, linear, homogeneous DE has characteristic roots*

$$-1 \pm 2i,\ -2,\ -2, -3 \pm 4i.$$

(a) *What is the order of the DE? (Notice the $\pm$ in the list of roots.)*

Solution: 6 roots implies it is a 6th order DE.

(b) *What is the general, real-valued solution.*

Solution: The 6 roots give 6 basic solutions:

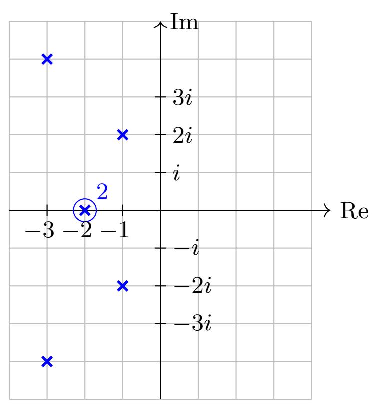
$$\begin{aligned}
x_1 &= e^{-t}\cos(2t) & x_2 &= e^{-t}\sin(2t) \\
x_3 &= e^{-2t} & x_4 &= te^{-2t} \\
x_5 &= e^{-3t}\cos(4t) & x_6 &= e^{-3t}\sin(4t)
\end{aligned}$$

The general solution is

$$x(t) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_6.$$

(c) *Draw the pole diagram for this system. Explain why it shows that all solutions decay exponentially to 0. What is the exponential decay rate of the general solution?*

Solution: For the pole diagram, we put an × at each root. We indicate the double root by circling it and putting a small 2 as a superscript.



Since all the poles are in the left half plane, all the basic solutions have negative exponents, i.e., decay exponentially to 0. This implies that all solutions, which are linear combinations of the basic ones, decay exponentially.

The decay rate is controlled by the right-most root. In this case, this has real part -1, so the general solution decays like e^{-t}.