ES.1803 Problem Section 7, Spring 2024

Problem 13.1. Compute the following by thinking of matrix multiplication as a linear combination of the columns of the matrix.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$

Problem 13.2. Is it a vector space? For all of these you just have to check that they are closed under addition and scalar multiplication, i.e. closed under linear combinations.

(a) The set of functions $f(x)$ such that $f(5) = 0$.

(b) The set of functions $f(x)$ such that $f(5) = 2$.

(c) The set of vectors (x, y) in the plane, such that $2x + 3y = 0$.

(d) The set of vectors (x, y) in the plane, such that $2x + 3y = 2$.

Problem 13.3. Convert the following ODE to a companion system: $x''' + 2x'' + 3x' + 4x = \cos(5t)$.

Problem 14.4. Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 4 & 6 & 2 & 4 \\ 3 & 6 & 10 & 3 & 6 \end{bmatrix}$. Put A in row reduced echelon form. Find the rank, a basis of the column space, a basis of the null space, and the dimension of each of the spaces.

Problem 14.5. Let $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Suppose R is the row reduced echelon form for A.

(a) What is the rank of A?

(b) Find a basis for the null space of A.

(c) Suppose the column space of A has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. Find a possible matrix for A. That is, give a matrix A with RREF R and the given column space.

(d) Find a matrix with the same row reduced echelon form, but such that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are in its column space.

Extra problems if time.

Problem 14.6. Suppose we want to solve $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

(a) When is this possible? Answer this in the form: "$\mathbf{b}$ must be a linear combination of the two vectors ..."

(b) $A\mathbf{x} = \mathbf{b}$ is certainly solvable for $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. (What is the obvious particular solution?) Describe the general solution to this equation, as $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$.

Problem 14.7. Suppose that the row reduced echelon form of the 4×6 matrix B is

$$R = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a linearly independent set of vectors of which every vector in the null space of B is a linear combination.

(b) Write the columns of B as $\mathbf{b_1}, \mathbf{b_2}, \dots, \mathbf{b_6}$. What is $\mathbf{b_1}$? What can we say about $\mathbf{b_2}$? Which of these vectors are linearly independent of the preceding ones? Express the ones which are not independent as explicit linear combinations of the previous ones. Describe a linearly independent set of vectors of which every vector in the column space of B is a linear combination.

Problem 14.8. Suppose we have a matrix equation

$$\begin{bmatrix} 1 & x & 2 \\ 3 & y & 4 \\ 5 & z & 6 \end{bmatrix} \mathbf{c} = \mathbf{0}$$

and all we know about the vector $\mathbf{c}$ is that $\mathbf{c} \neq \mathbf{0}$. What can we say about $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$?

Problem 14.9. For what values of y is it the case that the columns of $\begin{bmatrix} 1 & 1 & 2 \\ 3 & y & 4 \\ 5 & 1 & 6 \end{bmatrix}$ form a linearly independent set?

Problem 14.10. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$:

(a) Find the row reduced echelon form of A; call it R.

(b) The last column of R should be a linear combination of the first columns in an obvious way. This is a linear relation among the columns of R. Find a vector $\mathbf{x}$, such that $R\mathbf{x} = \mathbf{0}$, which expresses this linear relationship.

(c) Verify that the same relationship holds among the columns of A.

(d) Explain why the linear relations among the columns af R are the same as the linear relations among the columns of A. In fact, explain why, if A and B are related by row transformations, the linear relations among the columns of A are the same as the linear relations among the columns of B.

Problem 14.11. Consider the following system of equations:

$$\begin{aligned} x + y + z &= 5 \\ x + 2y + 3z &= 7 \\ x + 3y + 6z &= 11 \end{aligned}$$

(a) Write this system of equations as a matrix equation.

(b) Use row reduction to get to row echelon form. What is the solution set?

Problem 14.12. **(a)** Suppose we have a matrix equation

$$\begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & x \end{bmatrix}$$

Can you specify x? For any value of x you think is allowable, find such an equation. Can any of the $\bullet$'s be 0?

(b) Suppose we have a matrix equation

$$\begin{bmatrix} \bullet & 3 \\ \bullet & 4 \\ \bullet & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Can you specify the $\bullet$'s?

(c) Suppose we have a matrix equation

$$\begin{bmatrix} x & 3 \\ y & 4 \\ z & 5 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and all we know about the vector $\mathbf{c}$ is that $\mathbf{c} \neq \mathbf{0}$. What can we say about $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$?

Problem 14.13. Solve this system of linear equations. How many methods can you think of to solve this system?

$$\begin{aligned} x + y &= 5 \\ 3x + 2y &= 7 \end{aligned}$$

Problem 14.14. Solve the following equation using row reduction:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) At the end of the row-reduction process, was the last column pivotal or free? Is this related to the absence of solutions?

(b) Find a new vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ such that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has a solution.

Problem 14.15. Show that the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ corresponds to counter-clockwise rotation about the origin by 90 degrees, by computing the effect of this matrix on the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and drawing $\mathbf{v}_1$, $\mathbf{v}_2$, $A\mathbf{v}_1$, $A\mathbf{v}_2$ on the plane.

Problem 13.16. Make up a block matrix problem: Multiply a 4×4 matrix made up of four 2×2 blocks (two blocks of 0s, one block = identity, one block something else) times a 4×2 matrix with (i.e., two 2×2 blocks)