ES.1803 Problem Section 8, Spring 2024

Problem 17.1. **(a)** Let $A = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}$. Solve $\mathbf{x}' = A\mathbf{x}$.

(b) What is the solution to $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

(c) Decouple the system in Part (a). That is, make a change of variables that converts the system to a decoupled one. Write the system in the new variables.

Problem 16.2. **(a)** Find the eigenvalues and basic eigenvectors of $A = \begin{bmatrix} 3 & 1 & -3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$.

(b) Write A in diagonalized form.

(c) Compute A^5.

Problem 15.3. **(a)** Use row reduction to find the inverse of the matrix $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$.

(b) Use the record of the row operations to compute the determinant of A

Problem 16.4. Suppose $A = \begin{bmatrix} a & b & c \\ 0 & 2 & e \\ 0 & 0 & 3 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) For what value (or values) of a, b, c, e is A singular (non-invertible)?

(c) What is the minimum rank of A (as a, b, c, e vary)? What's the maximum?

(d) Suppose $a = -5$. In the system $\mathbf{x}' = A\mathbf{x}$, is the equilibrium at the origin stable or unstable.

Problem 16.5. Suppose that $A = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} S^{-1}$.

(a) What are the eigenvalues of A?

(b) Express A^2 and A^{-1} in terms of S.

(c) What would I need to know about S in order to write down the most rapidly growing exponential solution to $\mathbf{x}' = A\mathbf{x}$?

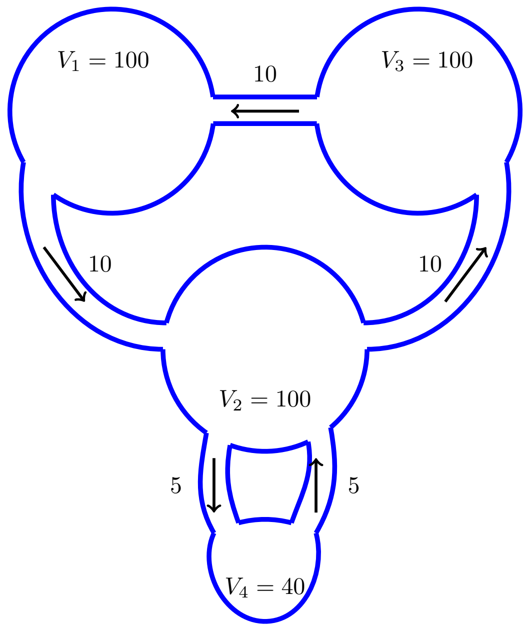
Problem 17.6. (**Complex roots**) Solve $\mathbf{x}' = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix} \mathbf{x}$ for the general real-valued solution.

Extra problems if time.

Problem 17.7. Solve $x' = -3x + y, \quad y' = 2x - 2y$.

Problem 16.8. **(b)** Find the eigenvalues and basic eigenvectors of $A = \begin{bmatrix} -3 & 4 \\ 2 & -5 \end{bmatrix}$.

Problem 17.9. The following figure shows a closed tank system with volumes and flows as indicated (in compatible units). Let's call the tank with $V_1 = 100$ tank 1, etc.



(a) Write down a system of differential equations modeling the amount of solute in each tank.

(b) Without computation you know one eigenvalue. What is it? What is a corresponding eigenvector?

(c) What can you say about all the other eigenvalues?

Problem 16.10. Suppose that the matrix B has eigenvalues 1 and 7, with eigenvectors

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

respectively.

(a) What is the solution to $\mathbf{x}' = Bx$ with $x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$?

(b) Decouple the system $\mathbf{x}' = B\mathbf{x}$. That is, make a change of variables so that system is decoupled. Write the DE in the new variables.

(c) Give an argument based on transformations why $B = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1}$ has the eigenvalues and eigenvectors given above.

Problem 16.11. Suppose the 2×2 matrix A has eigenvectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ with eigenvalues 2 and 4 respectively.

(a) Find $A(\mathbf{v_1} + \mathbf{v_2})$.

(b) Find $A(\mathbf{5v_1} + \mathbf{6v_2})$.

(c) Find $A \begin{bmatrix} 4 \\ 9 \end{bmatrix}$

Problem 16.12. **(a)** Without calculation, find the eigenvalues and and basic eigenvectors for $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

(b) Without calculation, find at least one eigenvector and eigenvalue for $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

Problem 16.13. Find the eigenvalues and basic eigenvectors of $A = \begin{bmatrix} -3 & 13 \\ -2 & -1 \end{bmatrix}$.

Problem 17.14. Solve the system $x' = x + 2y$; $\;y' = -2x + y$.

Problem 17.15. (**Repeated roots**) Solve $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}$.

Problem 15.16. Use row reduction to find inverses of the following matrices. As you do this, record the row operations carefully for later problems.

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -6 & 2 & -2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 2 & 2 \\ 3 & 5 & 7 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$

(d) $D = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 8 \\ 3 & 2 & 5 \end{bmatrix}$

Problem 15.17. Using just the record of the row operations in Problem 15.16 compute the determinant of each matrix.

Problem 15.18. Compute the transpose of the following matrices.

$$A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 6 & 7 \end{bmatrix}. \quad D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

Problem 15.19. Let $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

Show by direct computation that $(AD)^T = (D^T A^T)$.

Problem 15.20. **(a)** Recall the notation for inner product: $\langle \mathbf{v}, \mathbf{w} \rangle$. Assume $\mathbf{v}$ and $\mathbf{w}$ are column vectors. Write the formula for inner product in terms of transpose and matrix multiplication.

(b) Using this definition show $\langle A\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, A^T\mathbf{w} \rangle$.

You are not responsible for orthogonal matrices. The following is just for fun!

Problem 16.21.

(a) An orthogonal matrix is one where the columns are orthonormal (mutually orthogonal and unit length). Equivalently, S is orthogonal if $S^{-1} = S^T$.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Find an orthogonal matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$

(b) Decouple the equation $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.