ES.1803 Problem Set 1, Spring 2024

Part I (20 points)

Topic 1 (M, Feb. 5) Introduction to DEs; modeling; separable equations.
Read: Topic 1 notes.
Hand in: Part I problems: 1.1 - 1.6 (posted with psets).

Topic 2 (R, Feb. 8) Linear systems: input-response models.
Read: Topic 2 notes.
Hand in: Part I problems 2.1 - 2.3 (posted with psets).

Topic 3 (F, Feb. 9) Input-response models continued.
Read: Topic 3 notes.
Hand in: Part I problems 3.1 (posted with psets)

Part II (77 points)

Directions: Try each problem alone for 20 minutes. If, after this, you collaborate, you must write up your solutions independently. Consulting old problem sets is not permitted.

Problem 1 (Topic 1) (20: 10,10)
I have a bank account whose interest is compounded continuously with a variable interest rate $a(t)$. In 2023 I neither deposited nor withdrew money.

(a) Using the definition of deriviative as a limit of $\frac{\Delta x}{\Delta t}$, explain why the differential equation satisfied by the amount of money, $x(t)$ is $x'(t) = a(t)x(t)$.

(b) Much to my surprise, I discovered my balance rose only linearly. That is, $x(t) = mt + b$ for certain positive constants m and b. Is this possible? I suspect foul play and decide to apply the methods of 18.03. What can I conclude about the interest rate (in terms of m and b).

Problem 2 (Topic 3) (15: 10,5)
(a) This problem is warmup for Problem 3. It is about matching initial and final conditions at transition points of the input.

Solve the DE $x' + kx = f(t)$, $x(0) = 0$, where the input $f(t)$ is given by

$$f(t) = \begin{cases} 1 & \text{for } 0 \le t < 0.5 \\ 0 & \text{for } 0.5 \le t < 1 \\ 1 & \text{for } 1 \le t. \end{cases}$$

Give the solution $x(t)$ in *cases format*. (The function $f(t)$ above is given in cases format).

Use the format shown in the circuit example in the Topic 3 notes. For example, in the interval $0.5 \le t < 1$ the solution is $x(t) = C_1 e^{-k(t-0.5)}$, where $C_1 = x(0.5)$ matches the value of $x(t)$ at the end of the interval $0 \le t < 0.5$.

(b) For $k = 1$: graph the solution for Part (a). If this DE is modeling a mixing-tank situation, describe in words the behavior of the level of salt in the tank that this graph is showing. Aslo explain how the behavior of the response relates to the input salt rate $f(t)$.

Problem 3 (Topic 3) (22: 6,6,10)
A population of lemmings, crazed by global warming, has been flinging themselves into the sea at a rate faster than they can reproduce. As a result the deathrate of the lemmings is now greater than the birthrate, so the population is in decline. Studies show that if nothing is done, then the lemming population is halved every 3 years.

(a) Let x be the number of lemmings, t the time in years, $k > 0$ the decay rate of the lemming population.

(i) Modeling this with continuous variables, show that the DE for x as a function of t is $x' + kx = 0$. (We're just looking for a one line answer.)

(ii) Give the value of k, including units.

(iii) Assume that at time t_0 the population is x_0. Give the solution to the DE in (i) that satisfies this initial condition. (This gives the 'natural' behavior of the system, i.e., the population of lemmings, over time, when there is no input.)

(b) Alarmed by the potential loss of tourist business if the big annual lemming run should disappear, an importation/stocking program has been introduced by the Greenland Chamber of Commerce. For 6 months of the year (Jan. 1 to June 30), they import Canadian lemmings at a constant rate of r lemmings per year. For the other 6 months (July 1 to Dec. 31) the lemmings are left to their own devices.

(i) Show that $x' + kx = r$ is the DE that models the population when restocking is occurring. (Again, one line will suffice.)

(ii) Assume that, at time t_0, the population is x_0. Give the solution to the DE in (i) that satisfies this initial condition.

(c) Assume on Jan. 1 of 2023, the population was 1000. Also, assume the stocking rate r is 2000 lemmings per year. Using your answers to Parts (a) and (b), find the lemming population on Jan. 1, 2025. (For the purposes of this problem, you should take the period Jan. 1 to July 1 to be exactly 0.5 years and $k = 0.25$.)

So we all get the same answer, keep the full precision for intermediate values, but report your final answer as an integer. (It doesn't pay to contemplate fractional lemmings.)

Problem 4 (Topic 1) (20: 10,10)
Read Section 1.11 (orthogonal trajectories) in the notes for Topic 1.

For each of the following families of curves,

(i) find the ODE satisfied by the family

(ii) find the orthogonal trajectories to the given family

(iii) sketch the given family and the orthogonal trajectories.

(a) The family of curves $x = -y^2/2 + c$, where c is different for each member of the family.

(b) The family of curves $x^2 + 2y^2 = c$.

End of pset 1