ES.1803 Problem Set 2, Spring 2024 Solutions

Part II (95 points + 5 EC points)

Problem 1 (Topic 4) (15: 10,5)

For this problem you might want to first spend 5 minutes playing with the ***Complex Roots*** *applet at* ***https://mathlets.org/mathlets/complex-roots/***

(a) *For each of the following compute all three cube roots and plot them in the complex plane.*

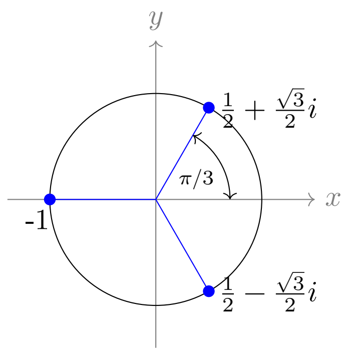
(i) -1 *(ii)* $1+\sqrt{3}i$

Solution: (i) We write -1 in polar form: $1 = e^{i(\pi+2\pi n)}$. Raising this to the $1/3$ power we get

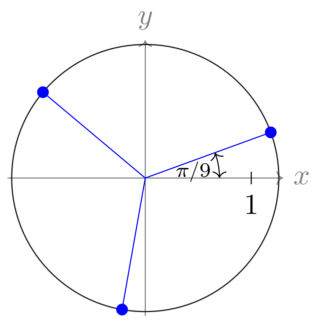
$$(-1)^{1/3} \;=\; e^{i\pi/3},\, e^{3\pi i/3},\, e^{5\pi i/3} \;=\; \boxed{-1,\, \frac{1}{2}+\frac{\sqrt{3}i}{2},\, \frac{1}{2}-\frac{\sqrt{3}i}{2}}.$$

(ii) We write $1+\sqrt{3}i$ in polar form $1+\sqrt{3}i = 2e^{i(\pi/3+2\pi n)}$. Raising this to the $1/3$ power we get

$$(1+\sqrt{3}i)^{1/3} \;=\; 2^{1/3}e^{\pi i/9},\, 2^{1/3}e^{7\pi i/9},\, 2^{1/3}e^{13\pi i/9}.$$



Cube roots of -1



Cube roots of $1+\sqrt{3}i$

(b) *Without computation (but with the applet if you like) describe the common pattern for* $(1)^{1/7}$ $(-1)^{1/7}$ *and* $(i)^{1/7}$. *(We're looking for a short simple answer.)*

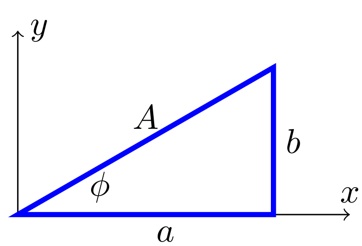
Solution: For all of them, the 7 roots are spaced evenly around the unit circle.

Problem 2 (Topic 4) (10)

The sinusoidal identity relates a sum of sinusoids in rectangular and polar (or amplitude phase) form:

$$a\cos(\omega t) + b\sin(\omega t) = A\cos(\omega t - \phi),$$

with a, b, A *and* ϕ *given in the figure below*



Verify this identity.

Solution: There are several ways to do this.

Method 1. Complex exponentials: Consider the product $(a-bi)(\cos(\omega t)+i\sin(\omega t))$ first in rectangular form and then in polar form. In rectangular form, we get

$$(a-bi)(\cos(\omega t)+i\sin(\omega t)) = a\cos(\omega t)+b\sin(\omega t)+i(a\sin(\omega t)-b\cos(\omega t)),$$

so that the real part is

$$\mathrm{Re}\left[(a-bi)(\cos(\omega t)+i\sin(\omega t))\right] = a\cos(\omega t)+b\sin(\omega t)$$

Now in polar form we have

$$a+bi = Ae^{i\phi} \text{ where } A=\sqrt{a^2+b^2} \text{ and } \phi = \mathrm{Arg}(a+bi) \quad \text{and} \quad \cos(\omega t)+i\sin(\omega t) = e^{i\omega t}.$$

So,

$$(a-bi)(\cos(\omega t)+i\sin(\omega t)) = Ae^{-i\phi}\cdot e^{i\omega t} = Ae^{i(\omega t-\phi)}.$$

The real part of this is $A\cos(\omega t-\phi)$. We now have the real part of the same product in two forms, which must be equal, i.e.

$$a\cos(\omega t)+b\sin(\omega t) = A\cos(\omega t-\phi),$$

which is what we wanted to show.

Method 2. Trig identities: The triangle shows that $a = A\cos\phi$ and $b = A\sin\phi$, so

$$a\cos(\omega t)+b\sin(\omega t) = A\cos\phi\cos(\omega t)+A\sin\phi\sin(\omega t) = A(\cos\phi\cos(\omega t)+\sin\phi\sin(\omega t)) = A\cos(\omega t-\phi)$$

The last equality above comes from a trig identity for cosines.

Problem 3 (Topic 5) (15: 5,5,5)

A linear, constant coefficient, homogeneous DE is called stable if all solutions go to 0 as t goes to infinity. Decide whether the following are stable. (Later we will connect this notion of stability to physical systems.)

(a) $x''+8x'+7x=0$

Solution: Characteristic equation: $r^2+8r+7=0$. So the roots are $r=-1,-7$.

General solution: $x(t) = c_1e^{-t}+c_2e^{-7t}$.

Since the exponents are negative, it is clear that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} c_1e^{-t}+c_2e^{-7t} = 0$$

for all values of the parameters c_1 and c_2. Thus the system $\boxed{\text{is stable.}}$

(b) $x''-9x=0$

Solution: Characteristic equation: $r^2-9=0$. So the roots are $r=3,-3$.

General solution: $x(t) = c_1e^{3t}+c_2e^{-3t}$.

Since the exponents are not both negative, it is clear that $x(t)$ goes to ∞ if $c_1>0$ and to $-\infty$ if $c_1<0$. In any case, not all solutions go to 0, so the $\boxed{\text{system is not stable.}}$

(c) $x'' + 2x' + 4x = 0$

Solution: Characteristic equation: $r^2 + 2r + 4 = 0$. So the roots are $r = -1 \pm \sqrt{3}\, i$.

General solution: $x(t) = c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(\sqrt{3}t)$.

Since the exponents are negative, it is clear that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(\sqrt{3}t) = 0$$

for all values of the parameters c_1 and c_2. Thus the system $\boxed{\text{is stable.}}$

Problem 4 (Topic 5) (30: 10,4,4,4,8)

Parts a-d of this problem deal with the equation $x^{(4)} - x = 0$.

(a) *Give the general* real-valued *solution to the equation.*

Solution: The characteristic equation is $r^4 - 1 = 0$. That is, the characteristic roots are the fourth roots of 1.

Since $1 = e^{2\pi n i}$ we have

$$r = e^{2\pi n i/4} = e^0, e^{\pi i/2},\ e^{\pi i},\ e^{3\pi i/2} = 1,\ i,\ -1,\ -i.$$

(Which we really knew without all the complex arithmetic just given.)

In order, the roots are 1, -1, $\pm i$. So the general solution to the DE is

$$x(t) = c_1 e^t + c_2 e^{-t} + c_3\cos(t) + c_4\sin(t).$$

(b) *Describe all the* real-valued *periodic solutions.*

Solution: From Part (a) we see that all periodic solutions are of the form $\boxed{x(t) = c_3\cos(t) + c_4\sin(t).}$ That is, the parameters c_1 and c_2 in the general solution must both be 0.

(c) *Describe all the solutions that go to 0 as $t \to \infty$.*

Solution: Again from Part (a): $x(t) = c_2 e^{-t}$. That is, the parameters c_1, c_3, and c_4 in the general solution must all be 0.

(d) *Describe the behavior of the general solution found in Part (a) as t goes to ∞.*

Solution: The general solution consists of an

– exponential that goes to ∞, i.e., the term $c_1 e^t$,

– a transient part, i.e., the term e^{-t} which goes to 0 as $t \to \infty$

– a sinusoidal part, i.e., the terms $c_3\cos(t) + c_4\sin(t)$.

Thus $\boxed{\text{as } t \to \infty \text{ most solutions go } \pm\infty.}$ In the unusal case where $c_1 = 0$, the solutions go asymptotically to a sinusoid

$$c_3\cos(t) + c_4\sin(t) = A\cos(t - \phi).$$

In very special cases, i.e., $c_1 = 0$, $c_3 = 0$ and $c_4 = 0$, $x(t)$ goes to 0.

(e) *Write down a third-order, constant coefficient, linear DE with the following properties:*

(i) Its characteristic polynomial, $P(r)$ has integer coefficients.

(ii) $P(r)$ has one real and two complex roots.

(iii) All solutions of the DE tend to 0 as $t \to \infty$.

Hint: start with the roots of the characteristic polynomial.

Solution: First pick roots. They need integer coefficients and negative real parts. Say, -1, -1+i, -1-i

Find the characteristic polynomial: $(r+1)(r+1-i)(r+1+i) = r^3 + 3r^2 + 4r + 2$.

Convert to a DE: $y''' + 3y'' + 4y' + 2y = 0$.

Problem 5 (Topic 5) **Damping** (25: 5,10,10,0,0)
An important use of damping is to bring a system into equilibrium. In many mechanical systems, vibrations are a noisy nuisance or even dangerous. For example, if your car hits a bump, or wind shakes a building, or your airplane wing starts to vibrate, then you want them to settle down promptly.

Consider our standard equation modeling a damped harmonic oscillator:

$$mx'' + bx' + kx = 0$$

By dividing by m and changing the letters used, we can write this in the form

$$x'' + 2\zeta\omega x' + \omega^2 x = 0.$$

Here, ζ (Greek zeta) is called the damping ratio and ω is called the natural frequency of the oscillator.

(a) *(i) Give the dimensions of ζ and ω.*

(ii) Solve the equation when $\zeta = 0$. Why is ω called the natural frequency of the spring?

Solution: (i) ζ is dimensionless! ω has dimension 1/time.

(ii) When $\zeta = 0$, the DE is $x'' + \omega^2 x = 0$. This is the model for a simple harmonic oscillator. (The roots are $\pm i\omega$). It has solution

$$\boxed{x(t) = c_1\cos(\omega t) + c_2\sin(\omega t).}$$

ω is called the natural frequency because it is the frequency of oscillation of the spring-mass alone, i.e., without damping.

(b) *Give the ranges of $\zeta \geq 0$ where the system is overdamped, underdamped, critically damped and undamped. Give the general real-valued solution in each case.*

Solution: The characteristic equation is $r^2 + 2\omega\zeta r + \omega^2 = 0$. This has roots

$$r = \omega\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right).$$

The type of damping is determined by whether the roots are real, complex, repeated.

Overdamped: real roots: $\zeta > 1$. This has solution

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

where $r_1 = \omega\left(-\zeta + \sqrt{\zeta^2 - 1}\right)$, $r_2 = \omega\left(-\zeta - \sqrt{\zeta^2 - 1}\right)$.

Underdamped: complex roots: $0 < \zeta < 1$. This has solution

$$x(t) = c_1 e^{at}\cos(\beta t) + c_2 e^{at}\sin(\beta t),$$

where $a = -\omega\zeta,\ \beta = \omega\sqrt{1-\zeta^2}$.

Critically damped: repeated roots: $\zeta = 1$. This has solution

$$x(t) = c_1 e^{rt} + c_2 t e^{rt},$$

where $r = -\omega\zeta$.

Undamped: No damping: $\zeta = 0$. Solution given in Part (a).

(c) *Now we are going to use an applet to explore how quickly each type of damping reaches equilibrium. Open and play with the applet*
https://web.mit.edu/jorloff/www/OCW-ES1803/zw-compare.html

Applet hints. *1. You can set the time by sliding the little vertical line on the t-axis or by clicking anywhere near the t-axis.*

2. Once the t-slider is selected you can use the arrow keys to move it one step at a time. By using the t-zoom, you can make the step size smaller.

3. The arrow keys work on any selected slider.

4. You can drag the graph left or right.

Equilibrium is when $x = 0$ and, at least according to our model, it always takes an infinite amount of time for the system to return to equilibrium. So, as a practical solution, let's say the system has 'reached practical equilibrium' once $|x(t)|$ is permanently less than 0.005. That is, once the solution reaches and stays within a small range of $x = 0$.

Now, set $\omega = 1.0,\ b_0 = 1,\ b_1 = 0.0$ *and set*

$$\zeta_1 = 0.8,\ \zeta_2 = 1.0,\ \zeta_3 = 1.1,$$

Here is a strategy for finding when the system reaches practical equilibrium. Zoom out to get a sense of where each graph reaches practical equilibrium. Then set the t-zoom to 1.0. As you get close, you should have the x-zoom set to around 0.01. Select the time slider and, using the arrow keys, find the time when each damping level reaches equilibrium. In order to avoid fussing too much, you should give your answer to 1 decimal place.

For each ζ_i, find the time when the solution reaches practical equilibrium.

Solution: According to the applet:
For $\zeta_1 = 0.8$, x 'reaches practical equilibrium' when $t \approx 7.2$.

For $\zeta_2 = 1.0$, x 'reaches practical equilibrium' when $t \approx 7.4$.

For $\zeta_3 = 1.1$, x 'reaches practical equilibrium' when $t \approx 9.1$.

(d) (optional, 0 points) *I love playing with this applet. So here's a challenge: With $\omega = 1.0,\ b_0 = 1.0,\ b_1 = 0.0$, find the value of the damping ratio ζ that causes x to reach practical equilibrium the fastest.*

Use the zoom levels and arrow keys to find the answer to 4 or 5 decimal places

Solution: According to the applet I get $\zeta = 0.87$ gives the fastest return to equilibrium. x reaches the given range when $t = 4.95255$. (When $\zeta = 0.86$ there is a point around $t = 6.06337$ where $x = 0.005$ on the applet.

Because the applet only gives ζ to two decimal places, the true answer is somewhere between 0.86 and 0.87.

(e) (optional, 0 points) *Check the 'Show roots' check box and play with the applet. This shows what is called the pole diagram for the system.*

How can you tell from the pole diagram if a system is oscillatory?

How can you tell from the pole diagram if a system is stable?

Solution: If the roots (also called poles) are in the left half-plane, i.e., real part negative, then the solution has negative exponents and the system is stable.

If the roots are on the real axis, the system is not oscillatory. If they have a nonzero imaginary part, then the solution contains sines and contains, so the system is oscillatory.

Problem 6 (Extra credit) (Topic 5) (5)
In this problem we consider the law of conservation of energy in a simple harmonic oscillator. We will show that the differential equation is consistent with this law. To be concrete, let's think about an undamped spring-mass system. Suppose we have mass m and spring constant k, then the DE modeling this system is

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{1}$$

Using your 8.01 knowledge, write the total energy of the system as kinetic + potential energy. Then use the DE to show that the total energy in the system is constant.

Solution: $\boxed{\text{Potential energy} = \frac{1}{2}kx^2, \quad \text{kinetic energy} = \frac{1}{2}mv^2 = \frac{1}{2}m(x')^2.}$

So: $\boxed{\text{Total initial energy } E = \frac{1}{2}kx^2 + \frac{1}{2}m(x')^2.}$

The rate energy is being dissipated is

$$\frac{dE}{dt} = kxx' + mx'x'' = x'(mx'' + kx) = 0.$$

The last equality follows from the DE in Equation 1. Thus the total energy is not changing, i.e., is constant, i.e., energy is conserved.

End of pset 2 solutions