ES.1803 Problem Set 4, Spring 2024 Solutions

Part II (110 + 10 EC points)

Problems 1-5 are really one long problem, aimed at filling out and understanding the table below. We will look at the following three DEs, modeling a damped spring-mass system driven in various ways. Our main focus will be on the amplitude response (or gain) and how it differs between the three systems.

System (i) $mx'' + bx' + kx = kf(t)$ *(system driven through the spring)*

System (ii) $mx'' + bx' + kx = bf'(t)$ *(system driven through the dashpot)*

System (iii) $mx'' + bx' + kx = m\omega^2 f(t)$ *(system driven by an unbalanced flywheel).*

In all cases the input is $f(t) = B\cos(\omega t)$, where B and ω are constants.

As usual, m, b and k are constants. The physical systems and the explanation of the models are given for (i) and (ii) in the Topic 6 notes. The system in (iii) is taken from the textbook by Edwards and Penney Section 2.6 (right before and in Problem 28).

Problem 1 (Topic 9) (15) **Output, Gain and Phase Lag**
As in Pset 3, Problem 2, let $f(t) = B\cos(\omega t)$ and $P(s) = ms^2 + bs + k$.

Either solve the DEs again or use your answers to Pset 3, Problem 2 to fill in the gain and phase lag rows in the table. For the gain, write your answer both formally, in terms of $|P(i\omega)|$, and in detail, in terms of m, b, k, and ω. For the phase lag, you just need to give the formal version in terms of $\mathrm{Arg}(P(i\omega))$. (But be careful, not every phase lag is simply $\mathrm{Arg}(P(i\omega))$.)

Show your work on your paper and either print out the chart or make a copy and put the final answer in that.

Now fill in the output row of the table. To make things simpler, you only need to write $x(t)$ in terms of $g(\omega)$ and $\phi(\omega)$.

Solution: See table. We took the solutions from Problem 2, Pset 3. Using these, we could fill in the output, gain and phase lag rows of the table.

	System (i)	System (ii)	System (iii)
Right-hand side of DE (Input = $f(t)$)	$kf(t)$	$bf'(t)$	$m\omega^2 f(t)$
Output $x(t)$	$Bg(\omega)\cos(\omega t - \phi(\omega))$	$Bg(\omega)\cos(\omega t - \phi(\omega))$	$Bg(\omega)\cos(\omega t - \phi(\omega))$
Gain $g(\omega)$	$\frac{k}{\lvert P(i\omega)\rvert} = \frac{k}{\sqrt{(k-mw^2)^2+b^2\omega^2}}$	$\frac{b\omega}{\lvert P(i\omega)\rvert} = \frac{b\omega}{\sqrt{(k-mw^2)^2+b^2\omega^2}}$	$\frac{m\omega^2}{\lvert P(i\omega)\rvert} = \frac{m\omega^2}{\sqrt{(k-mw^2)^2+b^2\omega^2}}$
Phase lag $\phi(\omega)$	$\text{Arg}(P(i\omega))$	$\text{Arg}(P(i\omega)) - \pi/2$	$\text{Arg}(P(i\omega))$
ω_r	$\sqrt{\frac{2mk-b^2}{2m^2}}$ (if $2mk - b^2 > 0$)	$\sqrt{\frac{k}{m}} = \omega_0$	$\frac{\sqrt{2}k}{\sqrt{2mk-b^2}}$ (if $2mk - b^2 > 0$)
Filter type	Low pass	Band pass	High pass

Table for Problems 1-5

Note. $P(i\omega) = k - m\omega^2 + ib\omega$, so

$$|P(i\omega)| = \sqrt{(k - m\omega^2)^2 + b^2\omega^2} \quad \text{and} \quad \text{Arg}(P(i\omega)) = \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right) \quad \text{in Q1 or Q2.}$$

Problem 2 (Topic 9) (30: 10,10,10) **Practical Resonance**

Hint: In all three parts of this problem, the maximum of $g(\omega)$ is at the same point as the minumum of $1/g(\omega)^2$, but the latter is easier to work with. Parts (a) and (c) require a small amount of calculus. Part (b) can be done with or without calculus.

(a) *Find the practical resonant frequency ω_r for System (i). Be sure to note the conditions when there is no resonance. Add ω_r to the table.*

Solution: Finding ω_r means finding the maximum of $g(\omega) = \dfrac{k}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$.

Following the hint: define $h(\omega) = \dfrac{1}{g(\omega)^2} = \dfrac{(k - m\omega^2)^2 + b^2\omega^2}{k^2}$.

Maximizing $g(\omega)$ is the same as minimizing $h(\omega)$.

Using calculus: $h'(\omega) = \dfrac{-4m\omega(k - m\omega^2) + 2b^2\omega}{k^2} = 0 \Rightarrow \omega = 0$ or $\omega = \sqrt{\dfrac{2mk - b^2}{2m^2}}$.

Therefore, $\boxed{\text{if } 2mk - b^2 > 0}$, there is a practical resonant frequency at $\boxed{\omega_r = \sqrt{\dfrac{2mk - b^2}{2m^2}}.}$

(In terms of the system's natural frequency: $\omega_r = \sqrt{\omega_0^2 - b^2/2m^2} < \omega_0$.)

(b) *Repeat Part (a) for System (ii).*

Solution: Using the same technique as in Part (a):

$$h(\omega) = \frac{1}{g(\omega)^2} = \frac{(k - m\omega^2)^2 + b^2\omega^2}{b^2\omega^2} = \frac{(k - m\omega^2)^2}{b^2\omega^2} + 1.$$

Because the first term is a square the minumum occurs when it is 0.

That is, the minimum is when $k - m\omega^2 = 0 \Rightarrow$ there is always a practical resonant frequency at $\boxed{\omega_r = \sqrt{k/m}.}$

(In terms of the system's natural frequency $\omega_r = \omega_0$.)

(c) *Repeat Part (a) for System (iii).*

Solution: As in Parts (a) and (b): $h(\omega) = \dfrac{1}{g(\omega)^2} = \dfrac{(k - m\omega^2)^2 + b^2\omega^2}{m^2\omega^4} = \dfrac{k^2}{m^2\omega^4} - \dfrac{2k}{m\omega^2} + 1 + \dfrac{b^2}{m^2\omega^2}$.

$$h'(\omega) = -\frac{4k^2}{m^2\omega^5} + \frac{4k}{m\omega^3} - \frac{2b^2}{m^2\omega^3} = 0$$

$$\Rightarrow -\frac{4k^2}{m^2} + \frac{4k}{m}\omega^2 - \frac{2b^2}{m^2}\omega^2 = 0 \Rightarrow \omega^2 = \frac{2k^2m^2}{2mk - b^2}.$$

$$\Rightarrow \boxed{w_r = \frac{\sqrt{2}k}{\sqrt{2mk - b^2}}, \text{ provided } 2mk - b^2 > 0.}$$

(Equivalently, divide numerator and denominator by $\sqrt{2}m$, $\omega_r = \dfrac{k/m}{\sqrt{k/m - b^2/2m^2}}$. Or in terms of the system's natural frequency $\omega_r = \dfrac{\omega_0^2}{\sqrt{\omega_0^2 - b^2/2m^2}}$.)

Problem 3 (Topic 9) (15: 10,5) **Visualization**

(a) *Start the applet:* `https://mathlets.org/mathlets/amplitude-and-phase-2nd-order/`. *This is a visualization of System (i). Turn on the Bode and Nyquist plots. Play with the applet, be sure you can identify which plot shows the amplitude response.*

Note: *The equation in the applet has $m = 1$, i.e., the coefficient of x'' is 1.*

Note: *When adjusting a slider, once you've selected the slider, you can get fine control using the arrow keys to move it. You can leave your mouse in the gain graph window to get a readout of the gain value.*

(i) Set $k = 1$ and $b = 0.5$. Use the graphs to determine if the system has a resonant frequency. If it does, give its value. Include a sketch of the amplitude response graph –no credit if you don't label the axes.

(ii) Set $k = 1$ and $b = 1.5$ and repeat Part (i)

(iii) Check your answers in (i) and (ii) against your answer to Problem 2.

(iv) Still with $k = 1$ find the value of b where practical resonance disappears. Check this with with your answer to Problem 2.

Solution: (i) There is practical resonance at $\omega_r \approx 0.95$. (Figure is below.)

(ii) No practical resonance. (Figure is below.)

(iii) In the applet, $m = 1$, so the formula for the resonant frequency in Problem 2 becomes $\omega_r = \sqrt{(2k - b^2)/2}$.

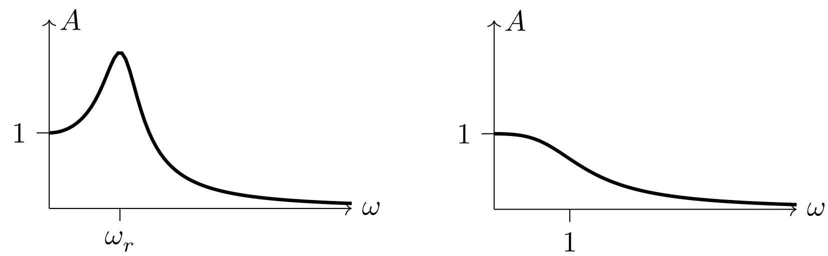
For $k = 1$, $b = 0.5$ we get $\omega_r = \sqrt{7/8} \approx 0.935$.

For $k = 1$, $b = 1.5$ we get $2k - b^2 = -0.25 < 0$, so no practical resonance.

Both of these match what we found in the applet in Parts (i) and (ii)

(iv) Using the applet we see practical resonance disappears when the amplitude curve does not have a maximum. If you watch the pixels carefully when $k = 1$ this happens in the applet when $b \approx 1.42$.

In general, there is no practical resonance when $2mk - b^2 < 0$. With $k = 1$, $m = 1$ this becomes $2 - b^2 < 0$. So there is not practical resonance for $\boxed{b > \sqrt{2} \approx 1.41}$. Again, the applet and the theory are aligned.



Amplitude for $b = 0.5$ and $b = 1.5$.

(b) *Start the applet:* `https://mathlets.org/mathlets/amplitude-and-phase-2nd-order-ii/`. *Check the Bode plots checkbox. Describe how the amplitude response changes as b and k are varied.*

Solution: The (practical) resonant frequency ω_r is independent of b (i.e., doesn't change as b changes). It increases as k increases and decreases to 0 as k goes to 0.

The peak amplitude is always $A = 1$. As b decreases the amplitude response becomes more spiked around ω_r.

Problem 4 (Topic 9) (15: 5,5,5) **Visualization and Filtering**

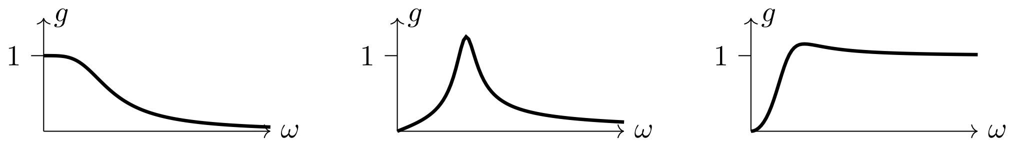
The systems in this pset can be considered filters. This means that they respond differently to input signals of different frequencies. For the frequencies where the gain is relatively large, we say the filter passes the frequency, for those where the gain is small, we say it stops the frequency. Here, we'll see how the power of ω in the gain affects the shape of the filter.

Start applet: `https://web.mit.edu/jorloff/www/OCW-ES1803/mbk4.html` *This applet shows graphs for the system $mx'' + bx' + kx = \omega^n \cos(\omega t)$, where n is a set-able parameter. So the* *applet covers all the powers of ω we've seen in the previous problems.*

(a) *Set $n = 0$, $m = 0.5$, $b = 1$, $k = 1$. Use the applet graphs to explain the use of the term 'low pass' in the filter row of the table.*

Solution: The three graphs below show the gain for Parts (a), (b) and (c).

The graph for Part (a) shows the gain starting at 1, staying level for a bit and then falling towards 0. This means that low frequencies are passed and high frequencies are stopped. Hence the name low pass filter.



Part (a): $n = 0$ Part (b): $n = 1$ Part (c): $n = 2$

(b) *Set $n = 1$, $m = 1.5$, $b = 0.8$, $k = 5$. Use the applet graphs to explain the use of the term 'band pass' in the filter row of the table. (A range of frequencies is referred to as a band.)*

Solution: The second graph above shows the gain has a peak and drops off to 0 on either side of the peak. This means the frequencies in a band around the peak frequency are passed and the others are stopped. Hence the name band pass filter.

(c) *Set $n = 2$, $m = 1$, $b = 1$, $k = 1$. Use the applet graphs to explain the use of the term 'high pass' in the filter row of the table.*

Solution: The third graph above shows the gain is small for low frequencies and is close to 1 for high frequencies. This means low frequencies are stopped and high frequencies are passed. Hence the name high pass filter.

Problem 5 (Topic 9) (20: 5,5,5,5) **AM Radio Tuning and LRC Circuits**
An LRC circuit can be modeled using the same DE as in system (ii). Specifically, we often want to know the voltage V_R across the resistor. This is modeled by the DE

$$LV_R'' + RV_R' + \frac{1}{C}V_R = RE'.$$

Where L = inductance in henries, R = resistance in ohms, C = capacitance in farads and E = input EMF in volts.

(a) *Assume $E = E_0\cos(\omega t)$ and use the table to give the periodic solution for V_R in amplitude-phase form.*

Solution: This is only a matter of translating the letters: $x \to V_R$, $m \to L$, $b \to R$, $k \to 1/C$, $B \to E_0$. So, $\boxed{V_R(t) = A\cos(\omega t - \phi(\omega))}$, where

$$A = \frac{E_0 R\omega}{|P(i\omega)|} = \frac{R\omega E_0}{\sqrt{(1/C - L\omega^2)^2 + (R\omega)^2}} = \frac{E_0}{\sqrt{\left(\frac{1-LC\omega^2}{RC\omega}\right)^2 + 1}}$$

and

$$\phi(\omega) = \text{Arg}(P(i\omega)) - \frac{\pi}{2} = \tan^{-1}\left(\frac{RC\omega}{1 - LC\omega^2}\right) - \frac{\pi}{2} \text{ in the Q1 or Q4.}$$

Note, since the phase lag can be in Q4 we can actually have the output V_R ahead of the input E (i.e., it has a negative phase lag).

(b) *Open the applet:* `https://web.mit.edu/jorloff/www/OCW-ES1803/lrc.html`. *This applet models an LRC circuit. The input voltage is a superposition of sine waves. Play with the applet –be sure to learn how to vary ω_1 and ω_2 by dragging the sliders on the amplitude plot.*

Describe what happens to the amplitude response plot as L, R and C are varied.

Solution: This is really the same as Problem 3(b): As L increases the amplitude peak moves to the left and the graph gets a little spikier.

As R decreases the peak doesn't move and the amplitude graph gets spikier.

As C increases the peak moves to the left.

(c) *An LRC circuit can be used as part of a simple AM radio tuner. In an AM radio broadcast the signal is given by $A\cos(\omega t)$ where ω is the 'carrier' frequency (between 530 and 1600 khz). To really carry information the amplitude A must vary with time –this is the amplitude modulation– but, we will ignore this right here.*

A typical range of values for this simple variable capacitor AM radio tuner is $L \approx 0.5$ microhenries, R is the resistance in the wire (very small) and C is between 0.02 and 0.2 microfarads. To keep things simple, we will use different ranges, however the idea is the same.

In the LRC Filter applet, set $\omega_1 = 1$ and $w_2 = 4$. Set the input amplitudes c_1 and c_2 to 1. Find settings for L, R and C so that the system filters out the ω_2 part of the signal i.e., the output looks (a lot) like a sine wave of frequency ω_1. Give your values for L, R and C.

Note. *Since the frequencies in the applet are not in the AM range, your values for L, R, C do not have to be in the same range as those in a typical variable capacitor tuner.*

Solution: One possibility is $L = 5.0$,z $C = 0.2$, $R = 0.33$. This gives $g(\omega_1) = 3.00$ and $g(\omega_2) = 0.05$, i.e., the gain at ω_1 is 60 times that of ω_2. In any case, we want $LC = 1$, so the peak gain is at ω_1. As R gets smaller, the ratio $g(\omega_1)/g(\omega_2)$ increases. In general, the smaller the value of R, the smaller the pass-band of the filter.

(d) *Since lots of stations are broadcasting at once, the antenna on your radio picks up a signal which is a superposition of lots of frequencies. The job of the tuner is to filter out all but the frequency you want. That is, the filter should pass a small band of frequencies around the desired one.*

Using the applet, set $L = 1$, $R = 0.5$. Now, vary C and then explain why a variable capacitor circuit could be used as an AM radio tuner.

Solution: As C varies the spike in the amplitude graph moves. That is, changing the band of frequencies that can pass through the filter. The job of the radio tuner is to pass the frequency of one radio station and stop all the others. So we vary C to 'tune' the circuit to the desired frequency.

For amusement: *check the 'N term mode' box. This changes the input to a sum of N sinusoids, with N set by a slider.*

Set $N = 1$, $L = 5.0$, $R = 0.32$, $C = 0.05$, $c = 0.2$ and $\omega = 2$. You should see a sinusoidal input and a sinusoidal output. Now increase N from 1 to 2 to 3, etc. The input changes as *more sinusoids are added. How does this change the output? Explain this in terms of the filter indicated by the gain graph.*

Solution: As N increases the output stays basically the same. The additional sinusoids in the input have frequency $4, 6, 8, \ldots$. The gain curve shows that these frequencies have a tiny gain. This means that the input at these frequencies produces output with a negligible amplitude.

Problem 6 (Topic 9) (15: 5,10,0)
There is another damped spring-mass model which can be used for further comparison and contrast, namely the automobile suspension system given in a previous problem. It is equivalent to a spring-mass system which is driven through both *the spring and dashpot*

$$mx'' + bx' + kx = kf(t) + bf'(t).$$

As before, we will take the input $f(t) = B\cos(\omega t)$.

(a) *Derive the formula for the amplitude response $g(\omega)$. As before, give the formal answer in terms of $|P(i\omega)|$ and $\mathrm{Arg}(P(i\omega))$ and the detailed answer in terms of m, b, k, ω.*

Solution: We complexify before taking the derivative:

$P(D)z = Bke^{i\omega t} + Bb\left(e^{i\omega t}\right)' = B(k + ib\omega)e^{i\omega t}, \quad x = \mathrm{Re}(z).$

ERF:

$$z_p(t) = \frac{B(k+ib\omega)e^{i\omega t}}{P(i\omega)} \Rightarrow x_p(t) = \frac{B|k+ib\omega|}{|P(i\omega)|}\cos(\omega t - \phi(\omega)).$$

Thus the gain is

$$g(\omega) = \frac{|k+ib\omega|}{|P(i\omega)|} = \frac{\sqrt{k^2+b^2\omega^2}}{\sqrt{(k-m\omega^2)^2+b^2\omega^2}}.$$

(b) *Derive the formula for the practical resonant frequency.*

Does practical resonance always occur in this case?

Solution: As before, we look for the minimum of

$$h(\omega) = \frac{1}{g(\omega)^2} = \frac{(k-m\omega^2)^2+b^2\omega^2}{k^2+b^2\omega^2} = 1 + \frac{-2km\omega^2+m^2\omega^4}{k^2+b^2\omega^2}.$$

Computing $h'(\omega)$ and setting it to 0 gives $\omega = 0$ or

$$m^2b^2\omega^4 + 2k^2m^2\omega^2 - 2k^3m = 0 \Rightarrow \boxed{\omega_r = \frac{\sqrt{-2k^2m^2 + \sqrt{4k^4m^4 + 8k^3m^3b^2}}}{\sqrt{2}mb}}.$$

Yes, since the term under the radical is always positive, the formula shows there is always a practical resonant frequency.

(c) *No question here, just a suggestion to look at the MIT mathlet:*

`https://mathlets.org/mathlets/amplitude-and-phase-2nd-order-iii/`

Solution: Nice animated visual of the spring-mass-dashpot-driver system.

Problem 7 (Topic 9) (Extra credit: 10: 5,5)

The complex gain for a system is the gain for the complexified system. For example, consider the system $P(D)x = kB\cos(\omega t)$, where $B\cos(\omega t)$ is the input. Using complex replacement, this becomes $P(D)z = kBe^{i\omega t}$. We can simplify this by writing

$$P(D)z = kBe^{st},$$

where s is any complex number. Of course, when we want to solve $P(D)x = kB\cos(\omega t)$, we take $s = i\omega$. The ERF says the solution to this is $\frac{kBe^{st}}{P(s)}$. If we consider the input to be Be^{st}, then the complex gain is $G(s) = \frac{k}{P(s)}$. (This is also known as the system or transfer function.)

The zero-pole diagram for a system is drawn in the complex plane. A pole for the complex gain is a (complex) value of s where the denominator of $G(s)$ has a 0. A zero is a value of s where $G(s) = 0$.

(a) *Draw a zero-pole diagram with a zero at $s = 0$ and poles at $s = -2$, $s = -1 \pm 2i$. Write down a system, specifying the input and output, that has this as its zero-pole diagram.*

Solution: The zero-pole diagram is below with the answer to Part (b). Using the roots we get the characteristic polynomial.

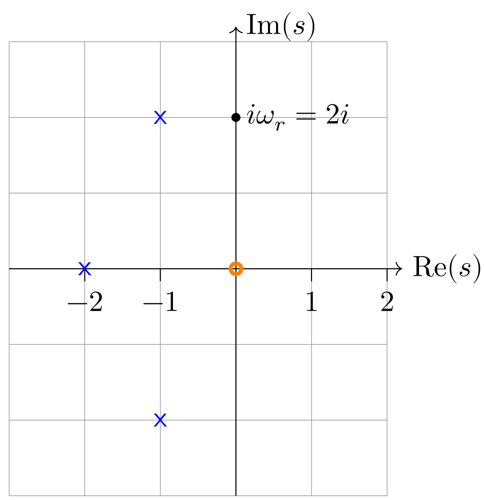
$$P(s) = (s + 2)(s + 1 - 2i)(s + 1 + 2i) = s^3 + 4s^2 + 9s + 10.$$

The system is of the form $P(D)x = Q(D)f$. The roots of Q are the zeros of the system, so $Q(s) = s$ works. A system with the specified zeros and poles is

$$(D^3 + 4D^2 + 9D + 10)x = Df.$$

(b) *This system has a practical resonant frequency. Indicate the approximate location of this on the pole diagram*

Solution: The resonant frequency is plotted on the imaginary axis. It should be near poles and away from zeros. The frequency $\omega = 2$, plotted at $i\omega = 2i$, is approximately the resonant frequency. (The actual resonant frequency is $\omega \approx 2.01229$.)



zero-pole diagram for Part (a), with resonant frequency at $2i$.

End of pset 4 solutions