ES.1803 Problem Set 6, Spring 2024 Solutions

Part II (139 points + 50 extra credit)

Problem 1 (Topic 13) (14: 2,2,2,2,2,2,2)

Go through Matlab Tutorial 1 from the class website. (If you prefer, you are welcome to use Julia. Julia has some differences with Matlab, so you may need to Google the syntax and install some packages.)

Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 1 & 0 & 0 \\ 1 & -4 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 0 \end{bmatrix}.$$

Use Matlab to compute all of the following. You may need to do a little googling to find the correct Matlab command.

(a) $A\mathbf{b}$

Solution: $\begin{bmatrix} 5 \\ 0 \\ -10 \\ 2 \end{bmatrix}$

(b) BA *(expect an error message).*

Solution: Error: Matrices are not compatible

(c) $\det(A)$.

Solution: $\det(A) = 19$

(d) A^{-1}.

Solution: $A^{-1} = \begin{bmatrix} 0.00000 & 0.00000 & -0.21053 & -0.05263 \\ 0.00000 & 0.00000 & -0.05263 & -0.26316 \\ 1.00000 & 0.00000 & 0.00000 & 0.00000 \\ 0.00000 & 1.00000 & 0.00000 & 0.00000 \end{bmatrix}$

(e) *The solution to* $A\mathbf{x} = \mathbf{b}$.

Solution: Use either the command `A\b` or `A`$^{(-1)}$`*b`

$\mathbf{x} = \begin{bmatrix} -1.05263 \\ -0.26316 \\ 2.00000 \\ 0.00000 \end{bmatrix}$.

(f) *The row reduced echelon form of* A *and* B

Solution: We use Matlab: `rref(A)` $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. (A square invertible matrix always has the identity for its row reduced echelon form.)

$\texttt{rref(B)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

(g) *A^T and B^T.*

Solution: We use the Matlab commands A', B'. (Of course, transpose is so easy we could do this by hand.)

Solution: $A^T = \begin{bmatrix} 0 & 0 & -5 & 1 \\ 0 & 0 & 1 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 2 & 2 \\ 3 & 1 & 0 \end{bmatrix}.$

Problem 2 (Topic 14) (25: 5,2,4,4,10)

(a) *Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 4 & 12 \end{bmatrix}$. By hand, find the reduced row echelon form for A. Track your steps carefully.*

Solution: $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 4 & 12 \end{bmatrix} \xrightarrow[R_3 = R_3 - 4R_1]{R_2 = R_2 - 2\,R_1} \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. (This is in row reduced echelon form.)

(b) *What is the rank of A?*

Solution: Rank = number of pivots = 1. (Also rank = dimension of column space.)

(c) *Give a basis for the column space of A.*

Solution: The pivot columns give a basis: $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.

(d) *Give a basis for the null space of A.*

Solution: A basis for the null space can be found using the RREF found in Part (a). A basis is $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

(e) *Give complete solutions for each of the following equations:*

(i) $A\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \qquad$ *(ii)* $A\mathbf{x} = \begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix}$

Solution: (i) By inspection we see that $\mathbf{x_p} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is a particular solution. From Part (d), the null space is the set of vectors of the form $c\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Thus the complete solution is $\begin{bmatrix} 2 \\ 0 \end{bmatrix} + c\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

(ii) The column space of A is $\left\{ c\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\}$. Since $\begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix}$ is clearly not in this column space, there are no solutions.

Problem 3 (Topic 14) (10)
Find a basis for the vector space of solutions to $x'' + 2x' + 2x = 0$.

Find a particular solution of $x'' + 2x' + 2x = 2$.

Find the general solution to $x'' + 2x' + 2x = 2$.

Solution: This is an old problem with some new language.

Characteristic equation: $r^2 + 2r + 2 = 0 \;\Rightarrow\;$ roots $= -1 \pm i$.

So the basic solutions are $e^{-t}\cos(t)$ and $e^{-t}\sin(t)$.

Guessing a constant solution, we see: $x_p(t) = 1$ is a particular solution.

The general solution is $x = 1 + c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$.

Problem 4 (Topic 15) (20: 5,5,5,5)
In the statements below A and B are square matrices of the same size. If the statement is true, give a reason (using the facts about determinants listed in the class notes for Topic 15). If it is false, provide a 2×2 counterexample.

(a) *If AB is invertible, then both A and B are invertible.*

(b) $\det(A - B) = \det A - \det B$.

(c) $\det(AB) = \det(BA)$.

(d) $\det(aB) = a\det(B)$, *where a is a number and aB means: B with all entries multiplied by a.*

Solution: **(a)** True.

Proof using determinants: AB is invertible means that $\det(AB) \neq 0$. Since $\det(AB) = \det(A)\det(B)$ this means $\det(A) \neq 0$ and $\det(B) \neq 0$. Since both determinants are non-zero both matrices are invertible.

You could also prove this by showing the null spaces A and B are both $\mathbf{0}$ or that their column spaces have full rank.

(b) False. Almost any A and B will show this, e.g., $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$. So, $\det(A - B) = 0$, but $\det(A) - \det(B) = 2 - 4 = -2$.

(c) True. $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$.

(d) False. For the 2×2 identity $\det(2I) = 4$, but $2\det(I) = 2$.

Problem 5 (Topic 17) (25: 5,10,5,5)
DE·SYSTEMS is an avant-garde pas de deux expressing in the universal language of dance the cycles of attraction and disdain of a pair of star-crossed armadillos named Armand (A) and Babette (B). Let $x = x(t)$ represent the time-varying level of A's attraction to B, and $y = y(t)$ represent the level of B's attraction to A, both in some suitable units. (Negative values of x or y will mean negative attraction, i.e., repulsion.) These being DE armadillos, we have, of course, equations giving the (coupled) rates of change of these levels of attraction,

namely the 2×2 *system of DEs:*

$$x' = x - y, \qquad y' = 2x - y.$$

(a) *Describe in words what this system of DEs is saying about the feelings of A and B for each other.*

Note: DE armadillos are known to be introspective, even to the point of brooding. (This is relevant to one of the two terms in each equation.)

Solution: Armand's story: Each term in $x' = x - y$ has a meaning.

$x' = x$ term: (A likes liking B) when A thinks about B, if he is liking B ($x > 0$), then his attraction has a positive slope and his attraction increases. If he is not liking B ($x < 0$) then he is more repelled, i.e., his attraction has a negative slope.

$x' = -y$ term: The more B likes A ($y > 0$), the less he likes her. The more she dislikes him ($y < 0$), the more he likes her. (He's just *that* kind of armadillo.)

Babette's story: Each term in $y' = 2x - y$ has a meaning.

$y' = -y$ term: B is the opposite of A in this respect. When she likes him ($y > 0$) then, as she thinks about him, she realizes that he's just not the armadillo she thought he was and likes him less. Whereas, when she's disliking him ($y < 0$) and thinks about him, she starts to like him better. (She's just *that* kind of armadillo.)

$y' = 2x$ term: B responds very positively (the factor of 2) when A likes her ($x > 0$) and very negatively when he doesn't ($x < 0$).

(b) *Take IC's* $x(0) = 2$, $y(0) = 2$. *Solve the system using matrix methods. Express your answers in "amplitude-phase" form.*

Solution: Coefficient matrix: $M = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

Eigenvalues (characteristic equation): $|M - \lambda I| = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$.

For eigenvectors we want a basis of the null space of $M - \lambda I$.

$\lambda = i$: $\quad (M - \lambda I) = \begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix}$. By inspection, we can take $\mathbf{v} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$.

(For $\lambda = -i$ we get the complex conjugate.)

We have a (complex-valued) solution

$$\mathbf{z}(\mathbf{t}) = e^{it} \begin{bmatrix} 1 \\ 1-i \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} 1 \\ 1-i \end{bmatrix} = \begin{bmatrix} \cos t + i \sin t \\ \cos t + \sin t + i(-\cos t + \sin t) \end{bmatrix}.$$

Both the real and imaginary parts of $\mathbf{z}$ are solutions. So the general real-valued solution is

$$\boxed{\mathbf{x}(t) = c_1 \begin{bmatrix} \cos t \\ \cos t + \sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ -\cos t + \sin t \end{bmatrix} = \begin{bmatrix} c_1 \cos t + c_2 \sin t \\ (c_1 - c_2)\cos t + (c_1 + c_2)\sin t \end{bmatrix}}.$$

or $\boxed{x(t) = c_1 \cos t + c_2 \sin t, \quad y(t) = (c_1 - c_2)\cos t + (c_1 + c_2)\sin t.}$

We use the initial conditions to find the coefficients:

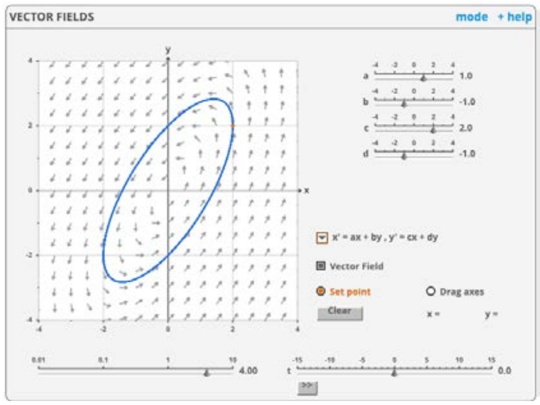
$$x(0) = c_1 = 2 \quad \text{and} \quad y(0) = c_1 - c_2 = 2 \Rightarrow c_2 = 0.$$

Thus, $\boxed{x(t) = 2\cos t, \quad y(t) = 2\cos t + 2\sin t = 2\sqrt{2}\cos(t - \pi/4).}$

(c) *Use the MIT Mathlet:* `https://mathlets.org/mathlets/vector-fields/`*. to obtain a graph of the solution to Part (b) in the* phase plane*. That is, the graph of y vs. x (the trajectory of the curve in the xy-plane). (You'll need to pick the right system from the drop-down menu.)*

Print out the phase plane picture and include it with your pset.

Solution:



Graph for Part (c)

(d) *Describe in words the unfortunate "minuet" A and B will be performing over time, which is given numerically and graphically by the pair of solutions $x = x(t)$, $y = y(t)$ to this DE system.*

Solution: The minuet takes us through all 4 quadrants, the direction indicates the sequence: A and B in love, B loves A but A hates B, they hate each other, A loves B but B hates A, back in love,...

Problem 6 (Topic 14) (15: 5,5,5) (Based on 3.3.4 #34 of Strang's *Linear Algebra*)
Suppose you know that the null space of the 3×4 matrix A has a basis consisting of the single vector $\begin{bmatrix} 1 \\ 4 \\ 2 \\ 0 \end{bmatrix}$.

(a) *What is the rank of A?*

Solution: A has 4 columns. The dimension of the null space is 1, so the dimension of the column space must be 3 (i.e., #pivots + #free = #columns). Thus the rank is 3.

(b) *What is the reduced echelon form of A?*

Solution: There is just one free variable. You can get a basis vector by setting the free variable equal to 1 and solving for the *earlier* pivotal variables. So the bottom nonzero entry in the basis vector must correspond to the free variable.

In our case, this means the 3rd column is free and Columns 1, 2 and 4 are pivotal. So, in reduced row echelon form, the matrix must by $\begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & * & 1 \end{bmatrix}$.

Since the 4th column is a pivot column, the bottom asterisk must be 0: $\begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

What's left is to find the values of the asterisks. We know that $\begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \\ 0 \end{bmatrix} = \mathbf{0}$.

Easy arithmetic now shows that the RREF of A is $\begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) *True or false: The equation $A\mathbf{x} = \mathbf{b}$ can be solved for any $\mathbf{b}$. Why?*

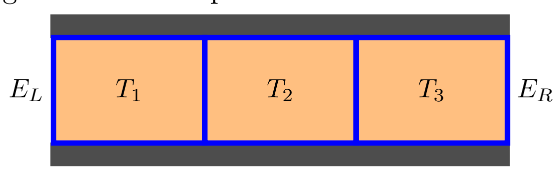
Solution: True. $\mathbf{b}$ is in $\mathbf{R}^3$. Since the rank is 3, the dimension of the column space is 3, i.e., the column space is all of $\mathbf{R}^3$.

Problem 7 (Topic 17) (20: 10,5,5)

Consider Example 13.7 in the class notes for Topic 13. This models the temperature in an inuslated bar. Assume that the temperatures E_L and E_R are both 0, i.e., the homogeneous case.

(a) *Call the coefficient matrix A. Find the eigenvalues of A. For each eigenvalue the corresponding eigenspace is one dimensional, find a non-zero eigenvector.*

Solution: Here's the figure from Example 13.7.



In matrix form the system of DEs is

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -2k & k & 0 \\ k & -2k & k \\ 0 & k & -2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

I computed the characteristic equation by augmenting $A - \lambda I$ and 'cross-hatching'. After some algebra it is

$$\det(A - \lambda I) = -(2k + \lambda)(\lambda^2 + 4k\lambda + 2k^2) = 0$$

This has roots $\lambda = -2k$, $(-2 \pm \sqrt{2})k$.

Finding basic eigenvectors (basis of $\text{Null}(A - \lambda I)$)

$\lambda_1 = -2k$: $(A - \lambda_1 I) = \begin{bmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{bmatrix}$. Putting this matrix in RREF, we get RREF $= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ So, a basic eigenvector is $\mathbf{v_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$\lambda_2 = (-2+\sqrt{2})k$: $\quad (A - \lambda_2 I) = \begin{bmatrix} -\sqrt{2}\,k & k & 0 \\ k & -\sqrt{2}\,k & k \\ 0 & k & -\sqrt{2}\,k \end{bmatrix}$. After some row reduction, we find $\mathbf{v_2} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$ is a basic eigenvector.

$\lambda_3 = (-2-\sqrt{2})k$: $\quad$ Similar calculations show $\mathbf{v_3} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$ is a basic eigenvector.

(b) *Describe the corresponding normal modes. Specify initial conditions leading to each one.*

Solution: Normal modes are

$$\mathbf{x_1}(t) = e^{-2kt}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{x_2}(t) = e^{(-2+\sqrt{2})kt}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \quad \mathbf{x_3}(t) = e^{(-2-\sqrt{2})kt}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}.$$

(Any non-zero multiples are also valid.)

The initial conditions are just the values at $t = 0$:

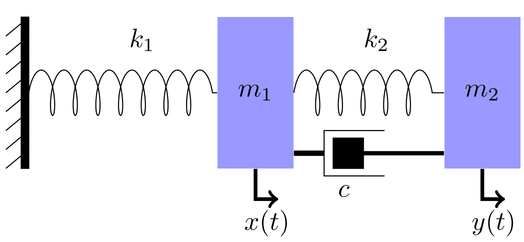
$$\mathbf{x_1}(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{x_2}(0) = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}, \quad \mathbf{x_3}(0) = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}.$$

(c) *Is this system stable?*

Solution: Yes, the system is stable because all the eigenvalues are negative.

Problem 8 (Topic 17) (10 + 5 EC: 5,5,EC-5)

(a) *Consider the unforced damped coupled spring system shown. The masses are m_1 and m_2; spring constants are k_1 and k_2; there is one damper with damping constant c; x is the displacement of m_1 from its equilibrium position and y is the displacement of m_2 from its equilibrium position. The damping force is proportional to the speed of the damper through its medium.*



Show that the system of DEs governing the behavior of the system is

$$\begin{aligned} m_1 x'' &= -k_1 x + k_2(y - x) + c(y' - x') \\ m_2 y'' &= -k_2(y - x) - c(y' - x'). \end{aligned}$$

Solution: The stretch of Spring 1 is x, so it exerts a force of $-k_1 x$ on m_1.

The stretch of Spring 2 is $y - x$, so it exerts forces of $k_2(y-x)$ and $-k_2(y-x)$ on m_1 and m_2 respectively.

The speed of the damper through its medium is $y' - x'$, so there are damping forces of $c(y'-x')$ and $-c(y'-x')$ on m_1 and m_2.

Adding the forces gives the equations we're supposed to show.

(b) *Find the companion system consisiting of 4 first-order equations. Write your answer in matrix form.*

Solution: Let $v = x'$, $w = y'$. The equations in Part (a) become

$$\begin{aligned} v' &= -\frac{k_1}{m_1}x + \frac{k_2}{m_1}(y-x) + \frac{c}{m_1}(w-v) \\ w' &= -\frac{k_2}{m_2}(y-x) - \frac{c}{m_2}(w-v) \end{aligned}$$

Putting these together with $x' = v$ and $y' = w$, we get the matrix equation

$$\begin{bmatrix} x' \\ y' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} - \frac{k_2}{m_1} & \frac{k_2}{m_1} & -\frac{c}{m_1} & \frac{c}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & \frac{c}{m_2} & -\frac{c}{m_2} \end{bmatrix} \begin{bmatrix} x \\ y \\ v \\ w \end{bmatrix}$$

(c) (Extra credit) *Now, let $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, $c = 1$.*

You probably don't want to go on to find the eigenvalues and eigenvectors by hand. Well, we can make Matlab do it for us by using the `[V , D] = eig(A)` *command.*

You can find basic instructions for Matlab and a short tutorial on eigenstuff on the class website. You can also learn about `eig` *by typing* `help eig` *at the Matlab prompt.*

Use this to find the real solutions for x and y. (Round to 2 decimal places in your answer.)

Solution: The coefficient matrix is $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 1 & -0.5 & 0.5 \\ 2 & -2 & 1 & -1 \end{bmatrix}$

The Matlab command `[V,D] = eig(A)` gives

$$V = \begin{bmatrix} 0.21 + 0.17\,i & 0.21 - 0.17\,i & 0.08 - 0.33\,i & 0.08 + 0.33\,i \\ -0.13 - 0.35\,i & -0.13 + 0.35\,i & -0.04 - 0.60\,i & -0.04 + 0.60\,i \\ -0.45 + 0.26\,i & -0.45 - 0.26\,i & 0.34 + 0.11\,i & 0.34 - 0.11\,i \\ 0.72 & 0.72 & 0.63 & 0.63 \end{bmatrix}$$

and

$$D = \begin{bmatrix} -0.68 + 1.81\,i & 0 & 0 & 0 \\ 0 & -0.68 - 1.81\,i & 0 & 0 \\ 0 & 0 & -0.07 + 1.03\,i & 0 \\ 0 & 0 & 0 & -0.07 - 1.03\,i \end{bmatrix}$$

The columns of V are the eigenvectors of A. The diagonal entries in D are the eigenvalues. (I actually used Octave, which has the same command. Julia will also work, but you'll have to use a search engine to find the correct commands.)

Since we only want solutions for x and y, we only need to use the first first 2 rows of our solutions. There are 4 complex modes: $e^{(-0.68+1.81i)t}\begin{bmatrix} 0.21+0.17i \\ -0.13-0.35i \end{bmatrix}$, $e^{(-0.07+1.03i)t}\begin{bmatrix} 0.08-0.33i \\ -0.04-0.60i \end{bmatrix}$, and their conjugates.

Using both real and imaginary parts of the above, the general, real-valued solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-0.68t}\begin{bmatrix} 0.21\cos(1.81t) - 0.17\sin(1.81t) \\ -0.13\cos(1.81t) + 0.35\sin(1.81t) \end{bmatrix} + c_2 e^{-0.68t}\begin{bmatrix} 0.17\cos(1.81t) + .021\sin(1.81t) \\ -0.35\cos(1.81t) - 0.13\sin(1.81t) \end{bmatrix}$$

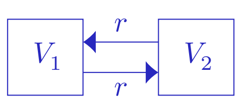
$$+ c_3 e^{-0.07t}\begin{bmatrix} 0.08\cos(1.03t) + 0.33\sin(1.03t) \\ -0.04\cos(1.03t) + 0.60\sin(1.03t) \end{bmatrix} + c_4 e^{-0.07t}\begin{bmatrix} -0.33\cos(1.03t) + 0.08\sin(1.03t) \\ -0.60\cos(1.03t) - 0.04\sin(1.03t) \end{bmatrix}$$

There are too many good problems to assign them all. Here are some more you can do for extra credit.

Extra credit problem 1 (Topic 17) (25: 5,5,5,10)

This problem examines closed circulating two and three compartment systems. We'll see that two compartment systems never oscillate, while oscillation is possible in three compartment systems.

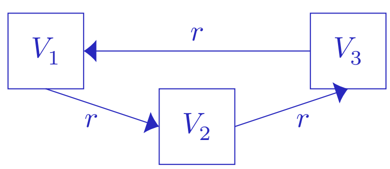
(a) *Consider the closed two-compartment system with flow rates and volumes shown. Let x_1 and x_2 be the amount of solute in Tanks 1 and 2 respectively. By analyzing input and output to each tank, derive (*but don't solve*) a system of DEs for this system*



Solution:

$$x_1' = -\frac{r}{V_1}x_1 + \frac{r}{V_2}x_2$$
$$x_2' = \frac{r}{V_1}x_1 - \frac{r}{V_2}x_2$$

(b) *By analyzing input and output to each tank, derive the DEs for the amount of salt in each tank for the closed three-compartment system with the general volumes and flow rates shown.*



Solution: We have the following notation and rates.

Amount of salt in Tank $i = x_i$.

Volume of solution in Tank $i = V_i$.

Rate salt leaves Tank i is $\dfrac{r}{V_i}x_i$.

Rate salt enters Tank 1 = rate it leaves Tank 2 = $\dfrac{r}{V_2}x_2$, etc.

Let $k_i = \dfrac{r}{V_i}$, then we have:

$$\begin{aligned} x_1' &= -k_1x_1 && && + k_3x_3 \\ x_2' &= k_1x_1 && - k_2x_2 && \\ x_3' &= && k_2x_2 && - k_3x_3. \end{aligned} \quad \text{or} \quad \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -k_1 & 0 & k_3 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & -k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(c) *Use physical reasoning to answer the following questions. What is the long-range behavior in the systems in Parts (a) and (b)? Which eigenvalue is responsible for this? Which eigenvalue controls how fast the system goes (asymptotically) to its equilibrium state?*

Solution: Eventually the tanks reach an equilibrium where all the concentrations are the same. The eigenvalue corresponding to this must be $\lambda = 0$. All the other eigenvalues must have negative real parts. (We could argue this more precisely as follows. At any point in time the tank with the greatest concentration has to be losing salt and the one with the minimum concentration has to be gaining salt. So the minimum concentration is increasing and the maximum is decreasing. Eventually they have to be the same.)

The rate is controlled by the eigenvalue with the biggest (least negative) real part.

(d) *(i) Show that the two-compartment system can never oscillate.*

(ii) Show (by finding an example with numbers) that the three-compartment system can *oscillate.*

Solution: (i) In the answer to Part (a) let $r/V_1 = k_1$ and $r/V_2 = k_2$.

Characteristic equation: $\lambda^2 + (k_1 + k_2)\lambda = 0 \Rightarrow$ eigenvalues $\lambda = 0, -(k_1 + k_2)$.

Since these are both real there can be no oscillation. (Note, we could have done this without solving the characteristic equation. We know in Part (c) that one eigenvalue was 0, so the other has to be real.)

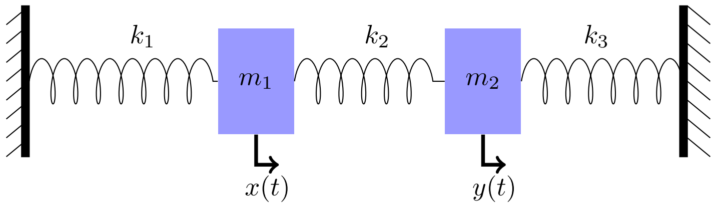
Solution: (ii) Characteristic equation: $\lambda^3 + (k_1 + k_2 + k_3)\lambda^2 + (k_1k_2 + k_1k_3 + k_2k_3)\lambda = 0$.

Eigenvalues: $\lambda = 0$ and $\lambda = \frac{-b+\sqrt{D}}{2}$, where $b = k_1 + k_2 + k_3$ and
$D = b^2 - 4(k_1k_2 + k_1k_3 + k_2k_3)$.

If $D < 0$ then there are complex eigenvalues $\Rightarrow$ oscillatory behavior.

Thus, if $k_1 = k_2 = k_3 = 1$, then $D = -3$ and the system will oscillate (as is settles down to equilibrium).

Extra credit problem 2 (Topic 17) (20: 5,5,5,5)
In this problem we'll look at a slightly different coupled spring-mass system.



We will use with the applet: `https://mathlets.org/mathlets/coupled-oscillators/`.

You should start by playing with it. It's pretty easy to figure out.

Now make sure the time t is set back to 0 and that the initial velocities are 0. (You can do this by switching on v_1 and v_2 and grabbing the end of the velocity indicator.) Set all the spring constants to $k_1 = k_2 = k_3 = 1$ and the masses to $m_1 = 2$, $m_2 = 1.25$.

(a) *In this system a normal mode is one where both x_1 and x_2 are sinusoids with the same frequency. Find two normal modes on the Mathlet. Is one in sync and one 180° out of sync, as they are in the equal mass case? In each case, use the crosshairs and readout of coordinates on the Mathlet to measure the amplitudes of the two sinusoids. In each case, write A_1 for the amplitude of the first mass and A_2 for the amplitude of the second, and compute the ratio A_2/A_1. Then measure the period of these sinusoids and record it.*

Solution: My strategy was to find initial settings with velocities 0 so that the two curves have the same zeros. Then the period is 4 times the first 0.

For the in phase normal mode I get: Period $= 8.08$, $A_1 = 1.26$, $A_2 = 1.0$ $A_1/A_2 = 1.26$.

For the 180° out of phase normal mode I found: Period $= 4.4$, $A_1 = 0.48$, $A_2 = 1.0$ $(x_2 = -1.0)$, $A_1/A_2 = 0.48$

(b) *Now write down the equations of motion ($m_1x'' = \cdots, m_2y'' = \cdots$) and the 4×4 "companion matrix." Write the companion matrix in block form where the upper right is the 2×2 identity.*

Solution: The equations are

$$\begin{aligned} m_1x'' &= -k_1x + k_2(y-x) \\ m_2y'' &= -k_2(y-x) - k_3y \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} 2x'' &= -2x + y \\ 1.25x_2'' &= x - 2y \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x'' &= -x + 0.5y \\ y'' &= 0.8x - 1.6y \end{aligned}$$

Let $v = x'$, $w = y'$ the equations become

$$\begin{aligned} x' &= v \\ y' &= w \\ v' &= -x + 0.5y \\ w' &= 0.8x - 1.6y \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x' \\ y' \\ v' \\ w' \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0.5 & 0 & 0 \\ 0.8 & -1.6 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ y \\ v \\ w \end{bmatrix} \Leftrightarrow \mathbf{x}' = \left[\begin{array}{c|c} 0 & I \\ \hline A & 0 \end{array}\right] \mathbf{x}$$

where $A = \begin{bmatrix} -1 & 0.5 \\ 0.8 & -1.6 \end{bmatrix}$

(c) *Find the eigenvalues using the trick in Part I Problem 17.1b. Based on these eigenvalues, what periods do the normal modes have? Compare with your measurements.*

Solution: In Problem 17.1b, we saw that the eigenvalues of the 'companion' matrix are $\pm$ the square root of the eigenvalues of A.

A has characteristic equation $\lambda^2 + 2.6\lambda + 1.2 = 0$.

This has roots -0.6 and -2.

So the companion matrix has eigenvalues $\pm\sqrt{0.6}\,i$ and $\pm\sqrt{2}\,i$.

The frequency of the normal modes are $\sqrt{0.6}$ and $\sqrt{2}$. Thus the periods are $2\pi/\sqrt{0.6} = 8.1116$ and $2\pi/\sqrt{2} = 4.4429$. These are nearly the same as the periods measured in Part (a).

(d) *Find the corresponding eigenvectors, write down the two normal modes. (I mean: write down the sinusoidal solutions for x and y.) Write them in the form $\mathbf{x}(t) = A\cos(\omega t - \phi)\mathbf{v}$, where $\mathbf{v}$ is a constant vector and A, ω are positive numbers, and ϕ can be anything.*

For each mode, let A_1 be the amplitude of x and A_2 that of y. Determine the ratio A_1/A_2 from this computation, and compare with your measurements.

Solution: First we find eigenvectors of A.

$\lambda_1 = -0.6$: $A - \lambda_1 I = \begin{bmatrix} -0.4 & 0.5 \\ 0.8 & -1 \end{bmatrix}$. We have an eigenvector $\mathbf{v_1} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$\lambda_2 = -2$: $A - \lambda_2 I = \begin{bmatrix} 1 & 0.5 \\ 0.8 & 0.4 \end{bmatrix}$. We have an eigenvector $\mathbf{v_2} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Problem Part I 17.1 showed that the eigenvalues and eigenvectors of the 4×4 companion matrix are given by

$$\mu_1 = \sqrt{0.6}\,i \quad : \quad \begin{bmatrix} \mathbf{v_1} \\ \mu_1 \mathbf{v_1} \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 5\sqrt{0.6}\,i \\ 4\sqrt{0.6}\,i \end{bmatrix} \qquad \mu_2 = -\sqrt{0.6}\,i \quad : \quad \begin{bmatrix} \mathbf{v_1} \\ \mu_2 \mathbf{v_1} \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -5\sqrt{0.6}\,i \\ -4\sqrt{0.6}\,i \end{bmatrix}$$

$$\mu_3 = \sqrt{2}\,i \quad : \quad \begin{bmatrix} \mathbf{v_2} \\ \mu_3 \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -\sqrt{2}\,i \\ 2\sqrt{2}\,i \end{bmatrix} \qquad \mu_4 = -\sqrt{2}\,i \quad : \quad \begin{bmatrix} \mathbf{v_2} \\ \mu_4 \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ \sqrt{2}\,i \\ -2\sqrt{2}\,i \end{bmatrix}$$

We take a complex mode and find its real and imaginary parts.

$$\mathbf{z}(t) = e^{\sqrt{0.6}\,it} \begin{bmatrix} 5 \\ 4 \\ \sqrt{0.6}\,i \\ 4\sqrt{0.6}\,i \end{bmatrix} = \begin{bmatrix} 5\cos(\sqrt{0.6}t) + i5\sin(\sqrt{0.6}t) \\ 4\cos(\sqrt{0.6}t) + i4\sin(\sqrt{0.6}t) \\ * \\ * \end{bmatrix}$$

We didn't bother with the entries for v and w because we only want formulas for x and y. Both the real and imaginary parts of $\mathbf{z}$ are solutions to our system. Using them and superposition, we have the normal mode

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 5\cos(\sqrt{0.6}t) \\ 4\cos(\sqrt{0.6}t) \end{bmatrix} + c_2 \begin{bmatrix} 5\sin(\sqrt{0.6}t) \\ 4\sin(\sqrt{0.6}t) \end{bmatrix}$$

Write $c_1\cos(\sqrt{0.6}\,t) + c_2\sin(\sqrt{0.6}\,t) = A\cos(\sqrt{0.6}\,t - \phi_1)$. Thus we have a normal mode:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\cos(\sqrt{0.6}\,t - \phi_1) \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

This shows that x and y are in phase and $A_1 = 5A$, $A_2 = 4A \Rightarrow A_1/A_2 = 5/4 = 1.25$. This agrees well with Part (a).

Similarly the other normal mode is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\cos(\sqrt{2}\,t - \phi_2)\begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

The minus sign in the vector shows that x and y are in 180° out of phase and $A_1 = A$, $A_2 = 2A \Rightarrow A_1/A_2 = 0.5$. This agrees well with Part (a).

End of pset 6 solutions