ES.1803 Problem Set 6, Spring 2024

Part I (56 points)

Topic 13 (F, Mar. 15) Linear algebra: vector spaces, matrices and linearity
Read: Topic 13 notes.
Hand in: Part I problems 13.1c, 13.2abcd, 13.3ab (posted with psets).

Topic 14 (M, Mar. 18) Linear algebra: row reduction, column spaces, null space
Read: Topic 14 notes.
Hand in: Part I problems 14.1ab, 14.2ab, 14.3abc (posted with psets).

Topic 15 (R, Mar. 21) Linear algebra: transpose, inverse, determinant
Read: Topic 15 notes.
Hand in: Part I problems 15.1acd (posted with psets).

Topic 16 (F, Mar. 22) Linear algebra: eigenvalues, diagonalization, decoupling
Read: Topic 16 notes.
Hand in: Part I problems 16.1a, 16.2a (posted with psets).

Topic 17 (T, Apr. 2) Matrix methods of solving systems, the companion matrix.
Read: Topic 17 notes.
Hand in: Part I problems 16.3a, 17.1a, 17.2abc, 17.3bc, 17.4, 17.5 (posted with psets).

Part II (139 points + 50 extra credit)

Directions: Try each problem alone for 20 minutes. If, after this, you collaborate, you must write up your solutions independently. Consulting old problem sets is not permitted.

Problem 1 (Topic 13) (14: 2,2,2,2,2,2,2)
Go through Matlab Tutorial 1 from the class website. (If you prefer, you are welcome to use Julia. Julia has some differences with Matlab, so you may need to Google the syntax and install some packages.)

Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 1 & 0 & 0 \\ 1 & -4 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 0 \end{bmatrix}.$$

Use Matlab to compute all of the following. You may need to do a little googling to find the correct Matlab command.

(a) $A\mathbf{b}$

(b) BA (expect an error message).

(c) $\det(A)$.

(d) A^{-1}.

(e) The solution to $A\mathbf{x} = \mathbf{b}$.

(f) The row reduced echelon form of A and B

(g) A^T and B^T.

Problem 2 (Topic 14) (25: 5,2,4,4,10)

(a) Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 4 & 12 \end{bmatrix}$. By hand, find the reduced row echelon form for A. Track your steps carefully.

(b) What is the rank of A?

(c) Give a basis for the column space of A.

(d) Give a basis for the null space of A.

(e) Give complete solutions for each of the following equations:

(i) $A\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$ (ii) $A\mathbf{x} = \begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix}$

Problem 3 (Topic 14) (10)
Find a basis for the vector space of solutions to $x'' + 2x' + 2x = 0$.

Find a particular solution of $x'' + 2x' + 2x = 2$.

Find the general solution to $x'' + 2x' + 2x = 2$.

Problem 4 (Topic 15) (20: 5,5,5,5)
In the statements below A and B are square matrices of the same size. If the statement is true, give a reason (using the facts about determinants listed in the class notes for Topic 15). If it is false, provide a 2×2 counterexample.

(a) If AB is invertible, then both A and B are invertible.

(b) $\det(A - B) = \det A - \det B$.

(c) $\det(AB) = \det(BA)$.

(d) $\det(aB) = a \det(B)$, where a is a number and aB means: B with all entries multiplied by a.

Problem 5 (Topic 17) (25: 5,10,5,5)
DE·SYSTEMS is an avant-garde pas de deux expressing in the universal language of dance the cycles of attraction and disdain of a pair of star-crossed armadillos named Armand (A) and Babette (B). Let $x = x(t)$ represent the time-varying level of A's attraction to B, and $y = y(t)$ represent the level of B's attraction to A, both in some suitable units. (Negative values of x or y will mean negative attraction, i.e., repulsion.) These being DE armadillos, we have, of course, equations giving the (coupled) rates of change of these levels of attraction, namely the 2×2 system of DEs:

$$x' = x - y, \qquad y' = 2x - y.$$

(a) Describe in words what this system of DEs is saying about the feelings of A and B for each other.

Note: DE armadillos are known to be introspective, even to the point of brooding. (This is relevant to one of the two terms in each equation.)

(b) Take IC's $x(0) = 2$, $y(0) = 2$. Solve the system using matrix methods. Express your answers in "amplitude-phase" form.

(c) Use the MIT Mathlet: https://mathlets.org/mathlets/vector-fields/. to obtain a graph of the solution to Part (b) in the *phase plane*. That is, the graph of y vs. x (the *trajectory* of the curve in the xy-plane). (You'll need to pick the right system from the drop-down menu.)

Print out the phase plane picture and include it with your pset.

(d) Describe in words the unfortunate "minuet" A and B will be performing over time, which is given numerically and graphically by the pair of solutions $x = x(t)$, $y = y(t)$ to this DE system.

Problem 6 (Topic 14) (15: 5,5,5) (Based on 3.3.4 #34 of Strang's *Linear Algebra*)
Suppose you know that the null space of the 3×4 matrix A has a basis consisting of the single vector $\begin{bmatrix} 1 \\ 4 \\ 2 \\ 0 \end{bmatrix}$.

(a) What is the rank of A?

(b) What is the reduced echelon form of A?

(c) True or false: The equation $A\mathbf{x} = \mathbf{b}$ can be solved for any $\mathbf{b}$. Why?

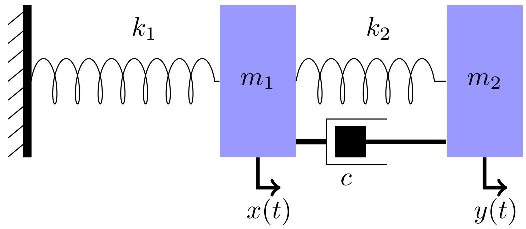
Problem 7 (Topic 17) (20: 10,5,5)
Consider Example 13.7 in the class notes for Topic 13. This models the temperature in an inuslated bar. Assume that the temperatures E_L and E_R are both 0, i.e., the homogeneous case.

(a) Call the coefficient matrix A. Find the eigenvalues of A. For each eigenvalue the corresponding eigenspace is one dimensional, find a non-zero eigenvector.

(b) Describe the corresponding normal modes. Specify initial conditions leading to each one.

(c) Is this system stable?

Problem 8 (Topic 17) (10 + 5 EC: 5,5,EC-5)
(a) Consider the unforced damped coupled spring system shown. The masses are m_1 and m_2; spring constants are k_1 and k_2; there is one damper with damping constant c; x is the displacement of m_1 from its equilibrium position and y is the displacement of m_2 from its equilibrium position. The damping force is proportional to the speed of the damper through its medium.



Show that the system of DEs governing the behavior of the system is

$$\begin{aligned} m_1 x'' &= -k_1 x + k_2(y - x) + c(y' - x') \\ m_2 y'' &= -k_2(y - x) - c(y' - x'). \end{aligned}$$

(b) Find the companion system consisiting of 4 first-order equations. Write your answer in matrix form.

(c) (Extra credit) Now, let $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, $c = 1$.

You probably don't want to go on to find the eigenvalues and eigenvectors by hand. Well, we can make Matlab do it for us by using the `[V , D] = eig(A)` command.

You can find basic instructions for Matlab and a short tutorial on eigenstuff on the class website. You can also learn about `eig` by typing `help eig` at the Matlab prompt.

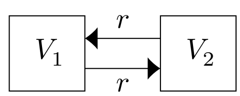
Use this to find the real solutions for x and y. (Round to 2 decimal places in your answer.)

(Extra credit problems on next page)

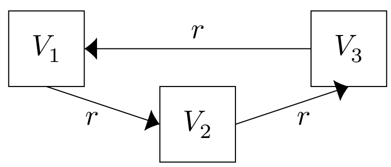
There are too many good problems to assign them all. Here are some more you can do for extra credit.

Extra credit problem 1 (Topic 17) (25: 5,5,5,10)
This problem examines closed circulating two and three compartment systems. We'll see that two compartment systems never oscillate, while oscillation is possible in three compartment systems.

(a) Consider the closed two-compartment system with flow rates and volumes shown. Let x_1 and x_2 be the amount of solute in Tanks 1 and 2 respectively. By analyzing input and output to each tank, derive (*but don't solve*) a system of DEs for this system



(b) By analyzing input and output to each tank, derive the DEs for the amount of salt in each tank for the closed three-compartment system with the general volumes and flow rates shown.

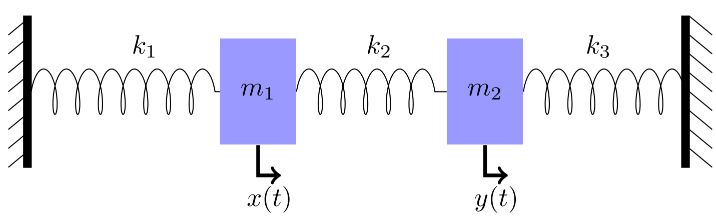


(c) Use physical reasoning to answer the following questions. What is the long-range behavior in the systems in Parts (a) and (b)? Which eigenvalue is responsible for this? Which eigenvalue controls how fast the system goes (asymptotically) to its equilibrium state?

(d) (i) Show that the two-compartment system can never oscillate.

(ii) Show (by finding an example with numbers) that the three-compartment system *can* oscillate.

Extra credit problem 2 (Topic 17) (20: 5,5,5,5)
In this problem we'll look at a slightly different coupled spring-mass system.



We will use with the applet: `https://mathlets.org/mathlets/coupled-oscillators/`.

You should start by playing with it. It's pretty easy to figure out.

Now make sure the time t is set back to 0 and that the initial velocities are 0. (You can do this by switching on v_1 and v_2 and grabbing the end of the velocity indicator.) Set all the spring constants to $k_1 = k_2 = k_3 = 1$ and the masses to $m_1 = 2$, $m_2 = 1.25$.

(a) In this system a normal mode is one where both x_1 and x_2 are sinusoids with the same frequency. Find two normal modes on the Mathlet. Is one in sync and one 180° out of sync, as they are in the equal mass case? In each case, use the crosshairs and readout of coordinates on the Mathlet to measure the amplitudes of the two sinusoids. In each case, write A_1 for the amplitude of the first mass and A_2 for the amplitude of the second, and compute the ratio A_2/A_1. Then measure the period of these sinusoids and record it.

(b) Now write down the equations of motion ($m_1x'' = \cdots, m_2y'' = \cdots$) and the 4×4 "companion matrix." Write the companion matrix in block form where the upper right is the 2×2 identity.

(c) Find the eigenvalues using the trick in Part I Problem 17.1b. Based on these eigenvalues, what periods do the normal modes have? Compare with your measurements.

(d) Find the corresponding eigenvectors, write down the two normal modes. (I mean: write down the sinusoidal solutions for x and y.) Write them in the form $\mathbf{x}(t) = A\cos(\omega t - \phi)\mathbf{v}$, where $\mathbf{v}$ is a constant vector and A, ω are positive numbers, and ϕ can be anything.

For each mode, let A_1 be the amplitude of x and A_2 that of y. Determine the ratio A_1/A_2 from this computation, and compare with your measurements.

End of pset 6