ES.1803 Problem Set 7, Spring 2024 Solutions

Part II (113 + 10 extra-credit points)

Problem 1 (Topic 20) (10)
Solve the initial value problem

$$2x''' + 12x'' + 22x' + 12x = \delta(t) + u(t)e^t, \ \ \textit{with rest initial conditions.}$$

Here $u(t)$ is the unit step function. This means that $u(t)e^t = \begin{cases} 0 & \textit{for } t < 0 \\ e^t & \textit{for } t > 0 \end{cases}$.

Hint: the characteristic roots are small negative integers.

Solution: The delta function changes the initial conditions at $t = 0$. So we have two cases:

Since $P(D)$ is the same in both cases, first we find the general homogeneous solution. Using the hint we find the characteristic roots are $r = -1, -2, -3$. So the homogeneous solution is

$$x_h(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}.$$

Case $t < 0$: $2x''' + 12x'' + 22x' + 12x = 0$; $x(0^-) = 0$, $x'(0^-) = 0$, $x''(0^-) = 0$.

This has solution: $x(t) = 0$.

Case $t > 0$: The delta function changes the rest pre-initial conditions to post-initial conditions. The post-initial conditions are

$$x(0^+) = x(0^-) = 0, \quad x'(0^+) = x'(0^-) = 0, \quad x''(0^+) = x''(0^-) + 1/2 = 1/2.$$

The DE in this case is

$$2x''' + 12x'' + 22x' + 12x = e^t. \tag{1}$$

A particular solution to Equation 1 is $x_p(t) = e^t/P(1) = e^t/48$.

Therefore, the general solution to Equation 1 is

$$x(t) = x_p(t) + x_h(t) = \frac{e^t}{48} + c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}.$$

We use the post-initial conditions to solve for c_j. The 3 equations are

$$\begin{matrix} c_1 + c_2 + c_3 + 1/48 & = 0 \\ -c_1 - 2c_2 - 3c_3 + 1/48 & = 0 \\ c_1 + 4c_2 + 9c_3 + 1/48 & = 1/2 \end{matrix} \Leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1/48 \\ -1/48 \\ 23/48 \end{bmatrix}$$

We solved this using row reduction: $c_1 = 1/8$, $c_2 = -1/3$, $c_3 = 3/16$. So,

$$\boxed{x(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{e^t}{48} + \frac{1}{8}e^{-t} - \frac{1}{3}e^{-2t} + \frac{3}{16}e^{-3t} & \text{for } t > 0. \end{cases}}$$

Problem 2 (Topic 20) (20: 5,5,5,5) **Lemmings really are adorable**
Back in your impulsive youth you helped a population of lemmings avoid extinction. But

your methods led to some tedious differential equations that no one liked solving. Now that you're older and wiser and truly understand impulses, you are ready to help the lemmings again.

Image from Wikimedia, In public domain. Also see Wikipedia: Lemming

Recall that the population y of lemmings is modeled by

$$y' + ky = f(t),$$

where t is measured in years, $k = 1.0$ is the growth rate and $f(t)$ is the input function.

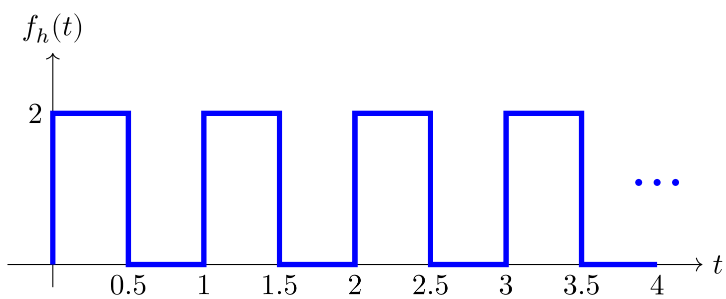
(a) *Your youthful input function could be described as a periodic box.*

$$f_h(t) = \begin{cases} \frac{1}{h} & \text{if } 0 < t < h \\ 0 & \text{if } h < t < 1 \\ \frac{1}{h} & \text{if } 1 < t < 1+h \\ 0 & \text{if } 1+h < t < 2 \\ \ldots . \end{cases}$$

(i) Graph this function for $h = 1/2$.

(ii) How many units of lemmings were added in each yearly cycle?

Solution: (i)



Periodic box $f_h(t)$ with $h = 1/2$.

(ii) The area under each box is 1, so 1 unit of lemmings were added each year.

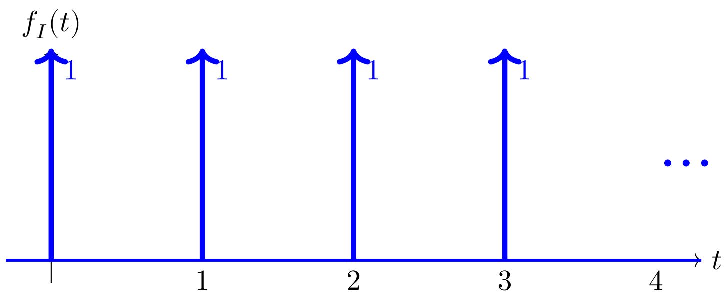
(b) *One problem with the input in Part (a) is that you had to spend half the year on the tundra. Another was that solving the DE with $f_h(t)$ as input took about half a year and almost made you quit ES.1803. So you decide to let h go to 0. That is, every year on January 1 you'll bring a truckload of lemmings (1 truckload = 1 unit of lemmings) to the wildlife reserve and release them all at once. (You may choose to stay a while and hope the Northern Lights are visible.)*

Call the new input function $f_I(t)$. Give its formula in terms of delta functions and sketch its graph. (Let $t = 0$ be January 1, 2023.)

Solution: As h goes to 0, the boxes get thinner and taller. The area remains one, so in the limit each box becomes a delta function.

$$f_I(t) = \delta(t) + \delta(t-1) + \delta(t-2) + \delta(t-3) + ...$$

Here is the graph. Each spike represents an impulse, the 1 next to each spike indicates the size of the impulse, i.e., the 'area' under the spike. (The function is 0 everywhere except at the spikes.)



Periodic impulse function, $f_I(t)$

(c) *In your absence the lemming population dwindled to nothing. So, when you started inputting lemmings, the intial population was 0. Solve the DE $y' + ky = f_I(t)$ with rest initial conditions.*

Solution: We show two methods of solving this. The first is to work one interval at a time using a series of pre and post-initial conditions. As usual, on each of the intervals between the spikes the input $f_I(t) = 0$.

On $t < 0$: DE is $x' + kx = 0$; IC $= x(0^-) = 0$.

The solution to this is easily found to be $\boxed{\text{on } t < 0,\ x(t) = 0.}$

At the end of this interval, we have $x(0^-) = 0$ (to be used in the next interval).

On $0 < t < 1$: DE is $x' + kx = 0$; The unit impulse at $t = 0$ gives the IC $= x(0^+) = x(0^-) + 1 = 1$.

The solution to this is easily found to be $\boxed{\text{on } 0 < t < 1,\ x(t) = e^{-kt}}$.

At the end of this interval, we have $x(1^-) = e^{-k}$ (to be used in the next interval).

On $1 < t < 2$: DE is $x' + kx = 0$; The unit impulse at $t = 1$ gives the IC $= x(1^+) = x(1^-) + 1 = e^{-k} + 1$.

The solution to this is easily found to be $\boxed{\text{on } 1 < t < 2,\ x(t) = (e^{-k} + 1)e^{-k(t-1)}.}$

At the end of this interval, we have $x(2^-) = (e^{-k} + 1)e^{-k} = e^{-2k} + e^{-k}$ (to be used in the next interval).

On $2 < t < 3$: DE is $x' + kx = 0$; The unit impulse at $t = 2$ gives the IC $= x(2^+) = x(2^-) + 1 = e^{-2k} + e^{-k} + 1$.

The solution to this is easily found to be $\boxed{\text{on } 2 < t < 3,\ x(t) = (e^{-2k} + e^{-k} + 1)e^{-k(t-2)}.}$

The pattern is now obvious!

$$x(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-kt} & \text{for } 0 < t < 1 \\ (e^{-k}+1)e^{-k(t-1)} & \text{for } 1 < t < 2 \\ (e^{-2k}+e^{-k}+1)e^{-k(t-2)} & \text{for } 2 < t < 3 \\ (e^{-3k}+e^{-2k}+e^{-k}+1)e^{-k(t-3)} & \text{for } 3 < t < 4 \\ \ldots \end{cases}$$

Our second method is to use the superposition principle. First we solve

$$x_n' + kx_n = \delta(t-n); \quad x(0^-) = 0$$

As usual, because of the impulse at $t = n$, we divide the problem into two intervals.

On $t < n$: DE is $x_n' + kx_n = 0$; IC is $x(0^-) = 0$.

The solution to this is easily found to be $\boxed{\text{on } t < n,\ x_n(t) = 0.}$

At the end of this interval, we have $x(n^-) = 0$ (to be used in the next interval).

On $n < t$: DE is $x' + kx = 0$; The unit impulse at $t = n$ gives the IC $= x(n^+) = x(n^-) + 1 = 1$.

The solution to this is easily found to be $\boxed{\text{on } n < t,\ x(t) = e^{-k(t-n)}}$.

Now, since the input $f_I(t) = \sum \delta(t-n)$ the sum of the x_n is a solution to the DE $x' + kx = f_I$. It is easy to check that this sum satisfies the rest initial conditions (since each term in it does). That is.

$$\begin{aligned} x(t) &= x_0(t) + x_1(t) + x_2(t) + \ldots \\ &= \begin{cases} 0 & \text{for } t < 0 \\ e^{-kt} + e^{-k(t-1)} & \text{for } 1 < t < 2 \\ e^{-kt} + e^{-k(t-1)} + e^{-k(t-2)} & \text{for } 2 < t < 3 \\ e^{-kt} + e^{-k(t-1)} + e^{-k(t-2)} + e^{-k(t-3)} & \text{for } 3 < t < 4 \\ \ldots \end{cases} \end{aligned}$$

You can check that both methods give the same answer!

(d) *Now we'll look at this graphically using a mathlet. Open* ***https://mathlets.org/mathlets/periodic-box/****.*

As usual, start the applet and familiarize yourself with its controls. Set $k = 1$ and $h = 0.5$.

(i) What happens to the response from rest as h goes to 0?

(ii) What happens to the impulse train response as t gets large?

Solution: (i) As $h \to 0$ the response from rest goes asymptotically to the impulse train response.

(ii) As t gets big: response from rest goes asymptotically to the periodic solution.

Problem 3 (Topic 21) (10: 5,5)

Without computing any integrals give the Fourier series for the following

(a) *$f(t) = \sin(t - \pi/4)$ (period 2π).*

Solution: $f(t) = \sin(t-\pi/4) = \sin(t)\cos(\pi/4) - \cos(t)\sin(\pi/4) = \boxed{\frac{\sqrt{2}}{2}\sin(t) - \frac{\sqrt{2}}{2}\cos(t).}$

(b) *$f(t) = sq(t - \pi/2)$, where $sq(t)$ is the odd square wave of period 2π and amplitude 1. (You should use the known series for $sq(t)$.)*

Solution: The Topic 21 notes give

$$sq(t) = \frac{4}{\pi}\sum_{n\ odd}\frac{\sin(nt)}{n} \quad \Rightarrow f(t) = \frac{4}{\pi}\sum_{n\ odd}\frac{\sin(n(t-\pi/2))}{n}.$$

This expression for $f(t)$ is good enough for everything we do in 18.03, but, to put it into official Fourier series form, we rewrite $\sin(n(t - \pi/2))$ as follows.

$$\sin(t-\pi/2) = -\cos(t), \quad \sin(3(t-\pi/2)) = \cos(3t)$$
$$\sin(5(t-\pi/2)) = -\cos(5t) \quad \sin(7(t-\pi/2)) = \cos(7t) \quad \text{etc.}$$

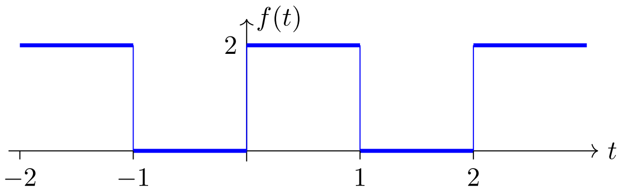
Thus, $\boxed{f(t) = \frac{4}{\pi}\left(-\cos(t) + \frac{\cos(3t)}{3} - \frac{\cos(5t)}{5} + \frac{\cos(7t)}{7} - \ldots\right)}$

Problem 4 (Topic 21) (28: 5,5,3,5,7,3)

(a) *Let $f(t)$ be the period 2 square wave, where, over one period, $f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ 0 & \text{if } 1 < t < 2 \end{cases}$*

Taking the jumps into account, compute the generalized derivative $h(t) = f'(t)$.

Solution: Here is the graph of $f(t)$



The regular part of f' is 0. Jumps in $f(t)$ lead to δ functions in its derivative. So,

$$h(t) = \ldots - 2\delta(t+1) + 2\delta(t) - 2\delta(t-1) + 2\delta(t-2) + \ldots = 2\sum_{n=-\infty}^{\infty}(-1)^n\delta(t-n)$$

(b) *Starting from the Fourier series for $f(t)$, find the Fourier series for $h(t)$.*

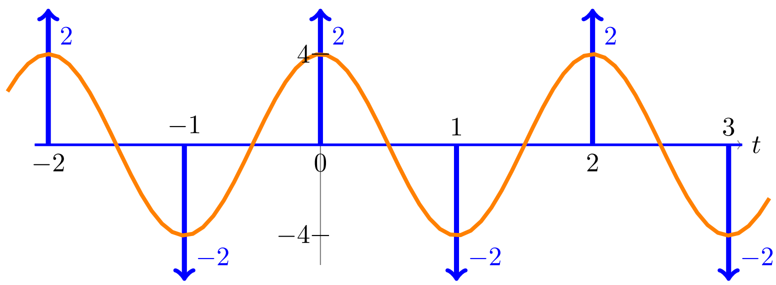
Solution: The known Fourier series for the odd, amplitude 1, period 2π square wave is $\frac{4}{\pi}\sum_{n\ odd}\frac{\sin(nt)}{n}$. Thus,

$$f(t) = 1 + \frac{4}{\pi}\sum_{n\ odd}\frac{\sin(n\pi t)}{n}.$$

Term-by-term differentiation gives $\boxed{h(t) = 4\sum_{n\ odd}\cos(n\pi t).}$

(c) *On the same axes, sketch the graphs of $h(t)$ and the first term in its Fourier series.*

Solution: Here are the graphs. The numbers next to the spikes indicate the 'area' under the spike.



(d) *Compute the coefficients of the Fourier series of $h(t)$ directly using the integral formulas and show they are the same as in the series found in Part (b).*

Solution: In Part (a) we found

$$h(t) = 2\sum_{n=-\infty}^{\infty} (-1)^n \delta(t-n).$$

To avoid worrying about endpoints for delta functions we'll compute the Fourier coefficients by integrating over the one period interval $(-1/2, 3/2)$. In this interval most of the terms in the series sum for $h(t)$ are zero. We have

$$h(t) = 2\delta(t) - 2\delta(t-1) \ \text{ for } \ -1/2 < t < 3/2.$$

The integrals are now easy to compute.

$$a_0 = \int_{-1/2}^{3/2} h(t)\,dt = \int_{-1/2}^{3/2} 2(\delta(t) - 2\delta(t-1))\,dt = 0.$$

$$\begin{aligned} a_n &= \int_{-1/2}^{3/2} h(t)\cos(n\pi t)\,dt = \int_{-1/2}^{3/2} 2(\delta(t) - 2\delta(t-1))\cos(n\pi t)\,dt \\ &= 2\cos(0) - 2\cos(n\pi) = 2 - 2(-1)^n = \begin{cases} 4 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even.} \end{cases} \end{aligned}$$

Likewise,

$$b_n = \int_{-1/2}^{3/2} h(t)\sin(n\pi t)\,dt = \int_{-1/2}^{3/2} 2(\delta(t) - 2\delta(t-1))\sin(n\pi t)\,dt = 2\sin(0) - 2\sin(\pi) = 0.$$

Thus, $h(t) = 4\cos(\pi t) + 4\cos(2\pi t) + \ldots = 4\sum_{n\,odd} \cos(n\pi t)$. (The same as in Part (b).)

(e) *Solve $Lx = x' + 9x = h(t)$.*

Solution: Replacing $h(t)$ by its Fourier series, we have to solve $x' + 9x = 4\sum_{n\,odd} \cos(n\pi t)$.

Characteristic polynomial: $P(r) = r + 9$. So,

$P(in\pi) = 9 + in\pi$; $|P(in\pi)| = \sqrt{81 + n^2\pi^2}$; $\text{Arg}(P(in\pi)) = \boxed{\phi(n) = \tan^{-1}(n\pi/9) \text{ in Q1.}}$

Individual terms: $x_n' + 9x_n = 4\cos(n\pi t)$. Using the SRF:

$$x_{n,p}(t) = \frac{4}{\sqrt{81 + n^2\pi^2}} \cos(n\pi t - \phi(n)).$$

Now, using superposition, we get

$$\begin{aligned} x_p(t) &= x_{1,p}(t) + x_{3,p}(t) + \ldots \\ &= \frac{4}{\sqrt{81 + \pi^2}} \cos(\pi t - \phi(1)) + \frac{4}{\sqrt{81 + 9\pi^2}} \cos(3\pi t - \phi(3)) + \ldots \\ &= \boxed{4 \sum_{n \text{ odd}} \frac{1}{\sqrt{81 + n^2\pi^2}} \cos(n\pi t - \phi(n)).} \end{aligned}$$

(f) *Does the solution in Part (e) contain any near resonant terms?*

Solution: No. The gain is a strictly decreasing function, so there are no resonant frequencies. Thus there are no terms in the solution that are near a resonant frequency. (There are never near resonant terms in the first-order case.)

Problem 5 (Topic 22) (15: 5,5,5)

Let $f(t) = \begin{cases} 2t^2 & \text{if } 0 \le t < \frac{1}{2} \\ 1 - t & \text{if } \frac{1}{2} \le t < 1 \\ 0 & \text{if } 1 \le t < 2. \end{cases}$

For each of the following, sketch the graph of its Fourier series over 3 full periods.

(a) $\tilde{f}_e(t)$, *the* even *periodic extension of* $f(t)$ *with period 4.*

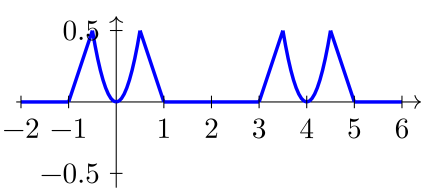
Solution: See below.

(b) $\tilde{f}_o(t)$, *the* odd *periodic extension of* $f(t)$ *with period 4.*

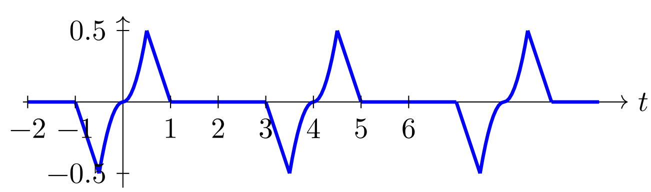
Solution: See below. (For Part (a) we only plotted 2 periods for space reasons.)

(c) $\tilde{f}(t)$, *the periodic extension of* $f(t)$ *with period 2.*

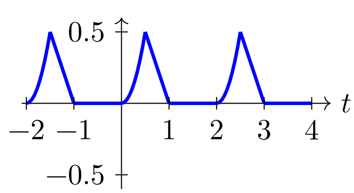
Solution: See below.



(a) Even extension



(b) Odd extension



(a) Periodic extension

Problem 6 (Topic 22) (30) *See the Fourier Sound lab exercise posted alongside this pset.*

Solution: See separate write up of the Fourier Sound lab exercise.

Extra credit 1 (Topic 17) (10)
(There isn't space for this problem. Since we don't want to interrupt the story, we've made it an extra credit problem.)

Continuing the story of Armand and Babette: Armand and Babette, exhausted from their seemingly endless cycles of attraction and repulsion, decide nevertheless to give it one more try, and so are off to Armadillo couples therapy. The Therapist admits that, given their current emotional patterns (which will be impossible to change quickly), it doesn't look good for a long-term stable happy relationship, but suggests a short-term external intervention to see if a break in the pattern might give them some time to work on the deeper issues. The Therapist therefore sends them to the Wizard, who concocts a special potion for them. Again let $x(t)$ and $y(t)$ denote the time-varying levels of A's attraction to B and B's attraction to A respectively. The "interaction coefficients" in the rate DEs for $x(t)$ and $y(t)$ are unchanged, since their emotional patterns are still the same; the effect of the Wizard's intervention on the rates of change of their feelings for each other is then to add the "external" functions $f_1(t)$ and $f_2(t)$ respectively to these DEs, so that we get

$$x' = x - y + f_1(t) \qquad y' = 2x - y + f_2(t).$$

To ease your computational load, we'll tell you that the corresponding homogeneous system has solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} \cos t \\ \cos t + \sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ -\cos t + \sin t \end{bmatrix}.$$

Whatever the Wizard intended, the effect on A and B of their scheduled ingestion of the potion turns out to be $f_1(t) = 3$ and $f_2(t) = -3$.

So that apparently *Armand has a good reaction and Babette a bad reaction to it.*

Guess a constant solution to find a particular solution to this inhomogeneous system.

What is the effect of the potion on A and B's situation? Is it positive or negative?

Solution: We have the equation $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 3 \\ -3 \end{bmatrix}$. We already know the homogeneous solution for this system and the problem only asks us to find a particular solution. Try the solution $\mathbf{x_p} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. Substitution gives $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \end{bmatrix}$.

Thus, $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = -\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$.

We have found a particular solution $\mathbf{x_p}(t) = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$.

The constant solution is an equilibrium. The homogeneous solution given above is purely oscillatory. So the general solution oscillates around this equilibrium. A positive equilibrium is better than their old oscillation around the equilibrium $(0, 0)$. As long as the oscillations aren't too large, they could stay in love forever. I would call the result positive!

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End of pset 7 solutions