ES.1803 Problem Section Problems for Quiz 2, Spring 2024 Solutions

Topic 4: Complex numbers and exponentials

Problem 4.1. *Make up and solve some simple algebra problems involving addition, subtraction, division, magnitude, complex conjugation.*

Solution: Provided by you!

Problem 4.2. (Polar coordinates)
Write $z = -1 + \sqrt{3}i$ in polar form.

Solution: Easily:

$$|z| = 2, \quad \text{and} \quad \boxed{\text{Arg}(z) = \phi = \tan^{-1}(-\sqrt{3}/1) = 2\pi/3}.$$

(We know ϕ is in the 2nd quadrant.) So, $\boxed{z = 2e^{i\phi} = 2e^{i2\pi/3}.}$

Problem 4.3. (Polar coordinates)
We know $-1 + \sqrt{3}\,i = 2e^{i2\pi/3}$. Use this to answer the following questions.

(a) *Compute the product $(-1 + \sqrt{3}i)(a + bi)$ (where a, b are real).*

Describe geometrically what multiplying by $-1 + \sqrt{3}i$ does.

Solution: $(-1 + \sqrt{3}i)(a + bi) = (-a - \sqrt{3}b) + (-b + \sqrt{3}a)i$.

In polar coordinates

$$(-1 + \sqrt{3}i)re^{i\theta} = 2e^{2\pi/3}re^{i\theta} = 2re^{i(\theta+2\pi/3)}.$$

Multiplying $z = a + bi$ by this number multiplies the magnitude of z by 2, and increases the argument by $2\pi/3$, i.e., it expands by a factor of 2 and rotates by 120°.

(b) *What are the polar coordinates of $(-1 + \sqrt{3}i)(a + bi)$ in terms of the polar coordinates of $a + bi = re^{i\theta}$?*

Solution: See answer to Part (a): The magnitude is $2r$ and the argument is $\theta + 2\pi/3$.

(c) *Describe the sequence of powers of $-1 + \sqrt{3}i$, positive and negative.*

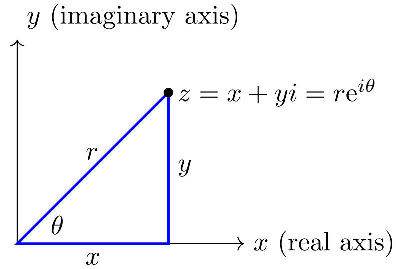
Solution: The powers of $-1 + \sqrt{3}i$ spiral out, rotating counterclockwise by 120° each time and growing by a factor of 2. Successive negative powers rotate clockwise by 120° and shrink by a factor of 1/2.

Problem 4.4. *Write $3e^{i\pi/6}$ in rectangular coordinates.*

Solution: By Euler's formula: $3e^{i\pi/6} = 3\cos(\pi/6) + 3i\sin(\pi/6) = \boxed{3\sqrt{3}/2 + i3/2.}$

Problem 4.5. (Trig triangle)
Draw and label the triangle relating rectangular with polar coordinates.

Solution:



Problem 4.6. *Compute $\dfrac{1}{-2+3i}$ in polar form. Convert the denominator to polar form first. Be sure to describe the polar angle precisely.*

Solution: In polar form $-2+3i = \sqrt{13}e^{i\theta}$, where $\theta = \arg(-2+3i) = \tan^{-1}(-3/2)$ in the second quadrant.

Therefore, $\dfrac{1}{-2+3i} = \dfrac{1}{\sqrt{13}e^{i\theta}} = \dfrac{1}{\sqrt{13}}e^{-i\theta}$.

Problem 4.7. *Find a formula for $\cos(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.*

Solution: First note, $\cos(3\theta) = \text{Re}(e^{3i\theta})$. We know,

$$e^{3i\theta} = (\cos(\theta) + i\sin(\theta))^3 = \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)$$

Taking the real part, we have $\boxed{\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta).}$

Problem 4.8. (Roots)
Find all fifth roots of -2. Give them in polar form. Draw a figure showing the roots in the complex plane.

Solution: We start by writing -2 in polar form, being sure to include all values of the argument:

$$-2 = 2e^{i\pi + i2n\pi}.$$

Raising this to the power 1/5 gives

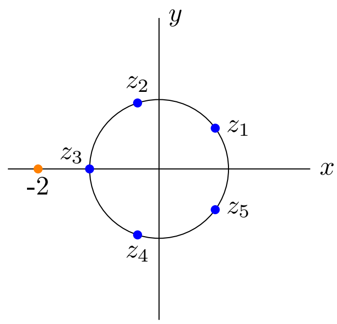
$$(-2)^{1/5} = 2^{1/5}e^{i\pi/5 + i2n\pi/5}.$$

Thus the 5 unique roots are:

$$z_1 = 2^{1/5}e^{i\pi/5}, \quad z_2 = 2^{1/5}e^{i3\pi/5}, \quad z_3 = 2^{1/5}e^{i5\pi/5}, \quad z_4 = 2^{1/5}e^{i7\pi/5}, \quad z_5 = 2^{1/5}e^{i9\pi/5}.$$

The only one of these that simplifies is $z_3 = 2^{1/5}e^{i5\pi/5} = -2^{1/5}$.

The figure below shows -2 and its fifth roots. Notice they are equally spaced around a circle of radius $2^{1/5}$.



Fifth roots of -2

Problem 4.9. (Complex replacement or complexification)
Compute $I = \int e^{4x}\cos(3x)\,dx$ using complex techniques.

Solution: Replacing $\cos(3x)$ by e^{i3x} we have: $I_c = \int e^{(4+3i)x}$, $I = \text{Re}(I_c)$.

Integrating: $I_c = \dfrac{e^{(4+3i)x}}{4+3i}$.

Polar form: $4+3i = 5e^{i\phi}$, where $\phi = \text{Arg}(4+3i) = \tan^{-1}(3/4)$ in Q1.

Thus, $I_c = \dfrac{e^{4x}}{5}e^{i(4x-\phi)}$. This implies

$$I = \text{Re}(I_c) = \frac{e^{4x}}{5}\cos(3x - \phi).$$

Problem 4.10. *The point of this problem is to help you distinguish between taking the real part of a function and finding which members of a* family of functions *are real-valued.*

(a) *Show the inverse Euler formulas are true:*

$$\cos(t) = (e^{it} + e^{-it})/2, \qquad \sin(t) = (e^{it} - e^{-it})/2i.$$

Solution: Use Euler's formula:

$$e^{it} = \cos(t) + i\sin(t)$$
$$e^{-it} = \cos(t) - i\sin(t)$$

Adding these two formulas gives $e^{it} + e^{-it} = 2\cos(t)$. Dividing by 2 then gives the inverse Euler formula for $\cos(t)$.

Likewise, subtracting the two formulas gives $e^{it} - e^{-it} = 2i\sin(t)$. Now, dividing by $2i$ gives the formula for $\sin(t)$.

(b) *Find all the real-valued functions of the form $\tilde{c}_1 e^{it} + \tilde{c}_2 e^{-it}$, where $\tilde{c}_1$ and $\tilde{c}_2$ are complex constants.*

(ii) Using Euler's formula we know that

$$\tilde{c}_1 e^{it} + \tilde{c}_2 e^{-it} = (\tilde{c}_1 + \tilde{c}_2)\cos(t) + i(\tilde{c}_1 - \tilde{c}_2)\sin(t)$$

If this is real-valued then the coefficients of $\cos(t)$ and $\sin(t)$ must be real:

$\tilde{c}_1 + \tilde{c}_2$ real implies $\mathrm{Im}(\tilde{c}_2) = -\mathrm{Im}(\tilde{c}_2)$.

$i(\tilde{c}_1 - \tilde{c}_2)$ real implies $\mathrm{Re}(\tilde{c}_2) = \mathrm{Re}(\tilde{c}_2)$.

Thus $\tilde{c}_1$ and $\tilde{c}_2$ are complex conjugates, say $\tilde{c}_1 = a - ib$ and $\tilde{c}_2 = a + ib$. Then

$$\tilde{c}_1 e^{it} + \tilde{c}_2 e^{-it} = 2a\cos(t) + 2b\sin(t)$$

Changing notation slightly, the answer is $\boxed{x(t) = a\cos(t) + b\sin(t).}$

Problem 4.11. *Find all the real-valued functions of the form $x = \tilde{c}e^{(2+3i)t}$.*

Solution: Let $\tilde{c} = a + ib$. Expanding x we get

$$x(t) = e^{2t}(a+ib)(\cos(3t) + i\sin(3t)) = e^{2t}(a\cos(3t) - b\sin(3t) + i(a\sin(3t) + b\cos(3t)))$$

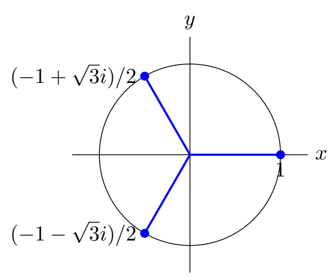
It's clear that the imaginary part can only be 0 if $a = b = 0$. So the only such real-valued function is $x(t) = 0$.

Problem 4.12. *Find the 3 cube roots of 1 by locating them on the unit circle and using basic trigonometry.*

Solution: We know one cube root is 1. This is on the unit circle and the three roots are evenly spaced around the circle. So the other two are at $e^{2\pi i/3}$ and $e^{4\pi i/3}$. Since $2\pi/3 = 120°$ and $4\pi/3 = 240°$, we can use our knowledge of 30, 60, 90 triangles to write the roots as

$$1, \qquad e^{2\pi i/3} = \frac{-1+\sqrt{3}i}{2}, \qquad e^{4\pi i/3} = \frac{-1-\sqrt{3}i}{2}$$

The figure below shows the three cube roots of 1.



Cube roots of 1

Problem 4.13. *Express in the form $a + bi$ the 6 sixth roots of 1.*

Solution: In polar form $1 = e^{i2\pi k}$, so

$$1^{1/6} = e^{i2\pi k/6} = e^{i\cdot 0},\ e^{i\pi/3},\ e^{i2\pi/3},\ e^{i3\pi/3},\ e^{i5\pi/3},\ e^{i5\pi/3}$$
$$= 1,\ \frac{1}{2} + i\frac{\sqrt{3}}{2},\ -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ -1,\ -\frac{1}{2} - i\frac{\sqrt{3}}{2},\ \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Problem 4.14. *Use Euler's formula to derive the trig addition formulas for* $\sin$ *and* $\cos$*.*

Solution: Use $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$.

$$\begin{aligned} e^{i\alpha}e^{i\beta} &= (\cos(\alpha) + i\sin(\alpha))(\cos(\beta) + i\sin(\beta)) \\ &= (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + i(\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)) \\ e^{i(\alpha+\beta)} &= \cos(\alpha+\beta) + i\sin(\alpha+\beta) \end{aligned}$$

Equating the two expressions above, we have:

$$(\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + i(\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)) = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

Equating the real and imaginary parts, we get the trig addition formulas:

$$\begin{aligned} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) &= \cos(\alpha+\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) &= \sin(\alpha+\beta). \end{aligned}$$

Problem 4.15. *Using the polar form, explain why* $|z^n| = |z|^n$ *and* $\arg(z^n) = n\arg(z)$ *for* n *a positive integer.*

Solution: In polar coordinates we have $z = re^{i\theta}$. So, $z^n = r^n e^{in\theta}$, i.e., $|z^n| = r^n = |z|^n$ and $\arg(z^n) = n\theta = n\arg(z)$. ■

Another way to say this is:

Magnitudes multiply, so $|z^n| = |z \cdot z \cdots z| = |z| \cdot |z| \cdots |z| = |z|^n$.

Arguments add, so $\arg(z^n) = \arg(z \cdot z \cdots z) = \arg(z) + \arg(z) + \cdots + \arg(z) = n\arg z$.

Problem 4.16. *Suppose* $z^n = 1$*. What must* $|z|$ *be? What are the possible values of* $\arg(z)$*, if* $z^n = 1$*?*

Solution: $|z|^n = 1$, and $|z| > 0$, so $|z|$ must be 1.

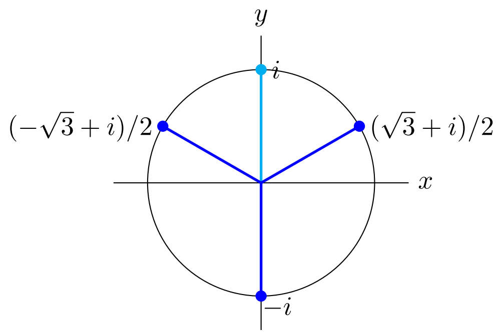
We must have $n\arg(z)$ is a multiple of 2π. So, $\arg(z) = 2m\pi/n$ for some integer n.

Problem 4.17. *Find the cube roots of* i*.*

Solution: We know that $i = e^{i\pi/2+2m\pi i}$, so the third roots are of the form $e^{i\pi/6+2m\pi i/3}$. The three unique roots are

$$e^{i\pi/6} = (\sqrt{3}+i)/2, \quad e^{i5\pi/6} = (-\sqrt{3}+i)/2, \quad e^{i9\pi/6} = -i.$$

The figure below shows the three cube roots of i.



Cube roots of i

Problem 4.18. *By using $(e^{it})^4 = e^{4it}$ and Euler's formula, find an expression for $\sin(4t)$ in terms of powers of $\cos(t)$ and $\sin(t)$,*

Solution: Compute

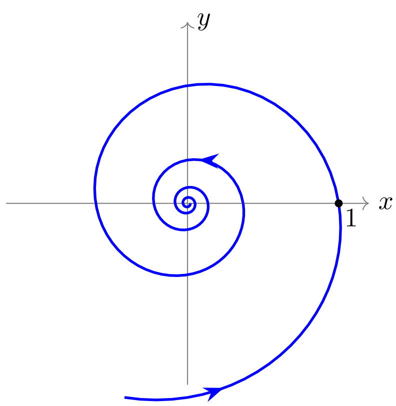
$$\begin{aligned} e^{4it} &= (e^{it})^4 \\ &= (\cos(t) + i\sin(t))^4 \\ &= \cos^4(t) + 4i\cos^3(t)\sin(t) - 6\cos^2(t)\sin^2(t) - 4i\cos(t)\sin^3(t) + \sin^4(t) \\ &= \left(\cos^4(t) - 6\cos^2(t)\sin^2(t) + \sin^4(t)\right) + i\left(4\cos^3(t)\sin(t) - 4\cos(t)\sin^3(t)\right) \end{aligned}$$

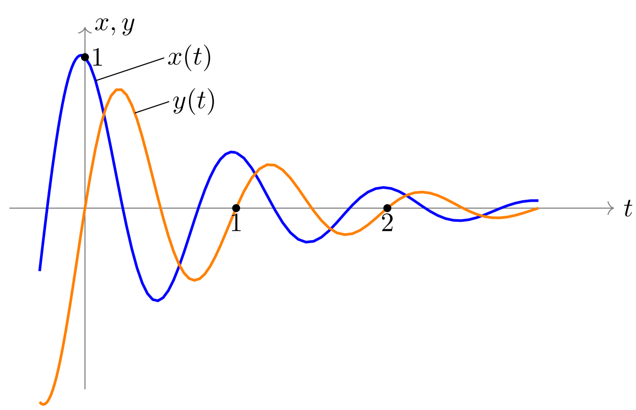
So, $\sin(4t) = \text{Im}(e^{4it}) = 4\cos^3(t)\sin(t) - 4\cos(t)\sin^3(t)$.

Problem 4.19. *Trajectories of $e^{(a+bi)t}$ can vary a lot, depending upon the value of the complex number $a + bi$. The "Complex Exponential" Mathlet shows this clearly. Invoke this applet if you can: **https://mathlets.org/mathlets/complex-exponential/**. You can use it to gain insight into the following questions.*

(a) *Sketch the trajectory of the complex-valued function $e^{(-1+2\pi i)t}$, and the graphs of its real and imaginary parts.*

Solution: This is a spiral moving towards the origin and turning counterclockwise. The real part is $e^{-t}\cos(2\pi t)$: a "damped sinusoid" with value 1 at $t = 0$. The imaginary part is $e^{-t}\sin(2\pi t)$: a damped sinusoid with value 0 at $t = 0$ and positive derivative there.





Left: Spiral in to origin. Right: Graphs of $x(t)$, $y(t)$.

(b) *For each of the following shapes, decide on all the values of $a+bi$ for which the trajectory of $e^{(a+bi)t}$ has this shape.*

(i) A circle centered at 0, traversed counterclockwise. What circles are possible?

(ii) A circle centered at 0, traversed clockwise.

(iii) A ray (straight half line) heading away from the origin.

(iv) A curve heading to zero as $t \to \infty$.

Solution: This will all depend upon Euler's formula

$$e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt))$$

Notice that $|e^{(a+bi)t}| = e^{at}$ and $\text{Arg}(e^{(a+bi)t}) = bt$.

(i) This can only happen if the magnitude is constant: so $a = 0$. To go counterclockwise, we must have $b > 0$. Ans: bi, $b > 0$: the "positive imaginary axis." The circle must be the unit circle.

(ii) Again $a = 0$, but now $b < 0$: the "negative imaginary axis."

(iii) Now b must be zero. For the magnitude to be increasing, we must have $a > 0$. Answer: real a, $a > 0$: the postive real axis.

(iv) For this we must have $a < 0$. b can be anything. So: $a + bi$ with $a < 0$: the left half plane.

Problem 4.20. **(a)** *Write $\cos(\pi t) - \sqrt{3}\sin(\pi t)$ in the form $A\cos(\omega t - \phi)$.*

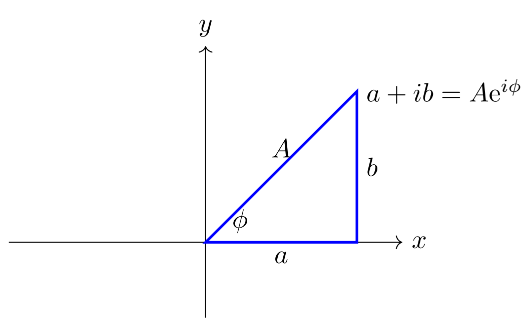
(b) *Write $5\cos\left(3t + \frac{3\pi}{4}\right)$ in the form $a\cos(\omega t) + b\sin(\omega t)$.*

(In each case, begin by drawing a right triangle with sides a and b, angle ϕ, hypotenuse A.)

Solution: This problem uses the identity

$$a\cos(\omega t) + b\sin(\omega t) = A\cos(\omega t - \phi)$$

in which (A, ϕ) are the polar coordinates of the coefficients (a, b).



(a) We have the point $(a, b) = (1, -\sqrt{3})$ (in the 4th quadrant). So, $A = \sqrt{1+3} = 2$ and $\phi = \tan^{-1}(-\sqrt{3}) = -\pi/3$. Thus,

$$\cos(\pi t) - \sqrt{3}\sin(\pi t) = 2\cos\left(\pi t + \frac{\pi}{3}\right).$$

(b) We have $A = 5$ and $\phi = 3\pi/4$. So,

$$a = 5\cos\left(-\frac{3\pi}{4}\right) = -\frac{5}{\sqrt{2}}, \qquad b = 5\sin\left(-\frac{3\pi}{4}\right) = -\frac{5}{\sqrt{2}}.$$

Thus, $5\cos\left(3t + \frac{3\pi}{4}\right) = -\frac{5}{\sqrt{2}}\cos(3t) - \frac{5}{\sqrt{2}}\sin(3t)$.

Problem 4.21. *Write $\cos(2t) + \sin(2t)$ in the form $A\cos(\omega t - \phi)$.*

Solution: The coefficients are $(a, b) = (1, 1)$, which have polar coordinates $A = \sqrt{2}$, $\phi = \frac{\pi}{4}$. So, $\cos(2t) + \sin(2t) = \sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)$.

Topic 5: Linear, constant coefficient, homogeneous DEs

Problem 5.22. **(a)** *Solve $x'' - 8x' + 7x = 0$ using the characteristic equation method.*

Solution: (Model solution) Characteristic equation: $r^2 - 8r + 7 = 0$.

Roots: $r = 7,\ 1$.

General real-valued solution: $x(t) = c_1 e^{7t} + c_2 e^{t}$.

(b) *Solve $x'' + 2x' + 5x = 0$ using the characteristic equation method.*

Solution: Characteristic equation: $r^2 + 2r + 5 = 0$.

Roots: $r = (-2 \pm \sqrt{4 - 20})/2 = -1 \pm 2i$.

General real-valued solution: $x(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$.

(c) *Assume the polynomial $r^5 + a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0$ has roots*

$$0.5, \quad 1, \quad 1, \quad 2 \pm 3i.$$

Give the general real-valued solution to the homogeneous constant coefficient DE

$$x^{(5)} + a_4 x^{(4)} + a_3 x^{(3)} + a_2 x'' + a_1 x' + a_0 x = 0.$$

Solution: Since we are given the roots, we can write the general solution directly:

$$x(t) = c_1 e^{0.5t} + c_2 e^{t} + c_3 t e^{t} + c_4 e^{2t}\cos(3t) + c_5 e^{2t}\sin(3t).$$

Problem 5.23. **(Unforced second-order physical systems)**
The DE $x'' + bx' + 4x = 0$ models a damped harmonic oscillator. For each of the values $b = 0, 1, 4, 5$ say whether the system is undamped, underdamped, critically damped or overdamped.

Sketch a graph of the response of each system with initial condition $x(0) = 1$ and $x'(0) = 0$. (It is not necessary to find exact solutions to do the sketch.)

Say whether each system is oscillatory or non-oscillatory.

Solution: The characteristic roots are $\dfrac{-b \pm \sqrt{b^2 - 16}}{2}$. We call the term under the square root the discriminant.

$b = 0$: The system is undamped and oscillatory (in fact sinusoidal).

$b = 1$: The discriminant $= 1 - 16 < 0$, so the roots are complex, which implies the system is underdamped and oscillatory.

$b = 5$: The discriminant is positive, so the roots are real, which implies system overdamped and non-oscillatory.

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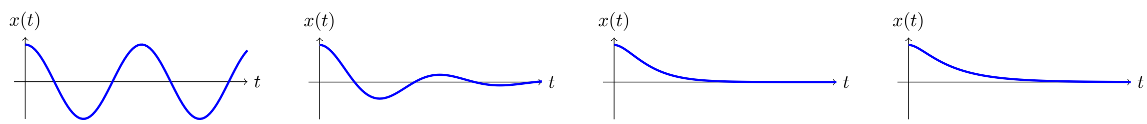
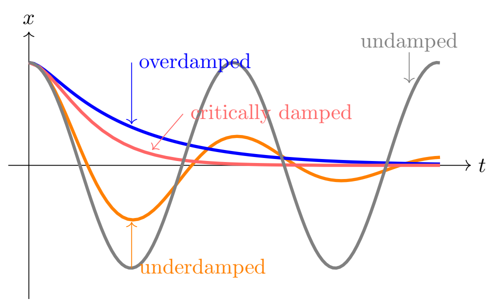
Here are plots of each of these solutions starting from $x(0) = 1$, $x'(0) = 0$.

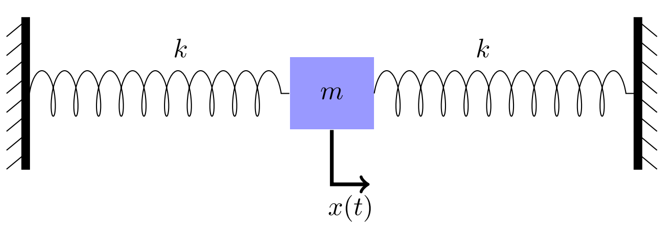


$b = 0$: undamped $\quad$ $b = 1$: underdamped $\quad$ $b = 4$: crit. damped $\quad$ $b = 5$: overdamped

The following figure is from the Topic 5 notes. It shows the different types of damping, though not necessarily using the coefficients in this problem. Note, that the initial conditions are all the same and, the initial velocity $x'(0) = 0$ causes all the graphs to have a horizontal tangent at $t = 0$.



Problem 5.24. *In the spring system below, both springs are unstretched when the position of the mass is $x = 0$, which is exactly in the middle. Write down a DE modeling the position of the mass over time.*



Solution: When the mass is at $x > 0$ then the left spring is stretched by x and the right spring is compressed by x, so the force on the mass is $m\ddot{x} = -2kx$ or $m\ddot{x} + 2kx = 0$.

Problem 5.25. *State and verify the superposition principle for $mx'' + bx' + kx = 0$, $(m, b, k$ constants).*

Solution: Superposition principle for linear, homogeneous DEs:

If x_1 and x_2 are solutions to the DE then so are all linear combinations $x = c_1x_1 + c_2x_2$.

Proof. Plug x into the DE and then chug through the algebra to show that x is a solution.

$$\begin{aligned} mx'' + bx' + kx &= m(c_1x_1 + c_2x_2)'' + b(c_1x_1 + c_2x_2)' + k(c_1x_1 + c_2x_2) \\ &= c_1mx_1'' + c_2mx_2'' + c_1bx_1' + c_2bx_2' + c_1kx_1 + c_2kx_2 \\ &= c_1 \underbrace{mx_1'' + bx_1' + kx_1}_{\substack{\text{0 by assumption that} \\ x_1 \text{ is a solution}}} + c_2 \underbrace{mx_2'' + bx_2' + kx_2}_{\substack{\text{0 by assumption that} \\ x_2 \text{ is a solution}}} \\ &= 0 \quad \blacksquare \end{aligned}$$

Problem 5.26. *A constant coefficient, linear, homogeneous DE has characteristic roots*

$$-1 \pm 2i, -2, -2, -3 \pm 4i.$$

(a) *What is the order of the DE? (Notice the $\pm$ in the list of roots.)*

Solution: 6 roots implies it is a 6th order DE.

(b) *What is the general, real-valued solution.*

Solution: The 6 roots give 6 basic solutions:

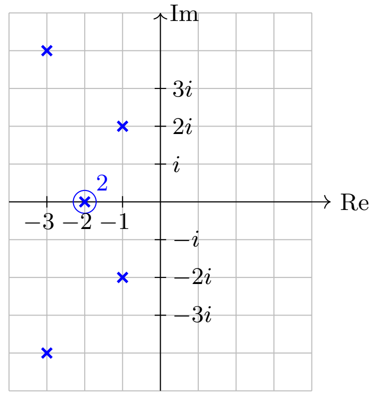
$$\begin{aligned} x_1 &= e^{-t}\cos(2t) & x_2 &= e^{-t}\sin(2t) \\ x_3 &= e^{-2t} & x_4 &= te^{-2t} \\ x_5 &= e^{-3t}\cos(4t) & x_6 &= e^{-3t}\sin(4t) \end{aligned}$$

The general solution is

$$x(t) = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_6.$$

(c) *Draw the pole diagram for this system. Explain why it shows that all solutions decay exponentially to 0. What is the exponential decay rate of the general solution?*

Solution: For the pole diagram, we put an x at each root. We indicate the double root by circling it and putting a small 2 as a superscript.



Since all the poles are in the left half plane, all the basic solutions have negative exponents, i.e., decay exponentially to 0. This implies that all solutions, which are linear combinations of the basic ones, decay exponentially.

The decay rate is controlled by the right-most root. In this case, this has real part -1, so the general solution decays like e^{-t}.

Topic 6: Operators, inhomogeneous DEs, ERF and SRF

Problem 6.27. *Let $P(D) = D^2 + 6D + 5I$. Find the general real-valued solution to each of the following.*

(a) $P(D)x = e^{-2t}$.

Solution: The characteristic polynomial is $P(r) = r^2 + 6r + 5$. Using the exponential response formula (ERF) to find a particular solution.

$P(-2) = -3$, so $\boxed{x_p(t) = \dfrac{e^{-2t}}{P(2)} = -\dfrac{e^{-2t}}{3}.}$

The characteristic roots are -1, -5, so the general homogeneous solution is

$$\boxed{x_h(t) = c_1 e^{-t} + c_2 e^{-5t}}.$$

By superposition, the general solution to the DE is $\boxed{x(t) = x_p(t) + x_h(t) = -\dfrac{e^{-2t}}{3} + c_1 e^{-t} + c_2 e^{-5t}.}$

(b) $P(D)x = \cos(3t)$.

Solution: We'll use the sinusoidal response formula (SRF) to find a particular solution $x_p(t)$.

$P(3i) = -4 + 18i$. So, $|P(3i)| = 2\sqrt{85}$ and $\boxed{\phi = \text{Arg}(P(3i)) = \tan^{-1}(-9/2) \text{ in Q2}}$.

Thus the SRF gives $\boxed{x_p(t) = \dfrac{\cos(3t - \phi)}{|P(3i)|} = \dfrac{\cos(3t - \phi)}{2\sqrt{85}}.}$

Using the homogeneous solution from Part (a), the general solution to the linear DE is $\boxed{x(t) = x_p(t) + x_h(t).}$

Alternatively, we could have used complex replacement with the DE $P(D)z = e^{3it}$ –see the answer to Part (c)

(c) $P(D)x = e^{2t}\cos(3t)$.

Solution: We start by using complex replacement to get the equation:

$$P(D)z = e^{(2+3i)t}, \text{ with } x = \text{Re}(z).$$

The ERF gives a particular solution $z_p(t) = \dfrac{e^{(2+3i)t}}{P(2+3i)}$.

Computing: $P(2+3i) = 12+30i$, so $|P(2+3i)| = 6\sqrt{29}$ and $\boxed{\phi = \text{Arg}(P(2+3i)) = \tan^{-1}(5/2) \text{ in Q1.}}$
Therefore,

$$z_p(t) = \frac{e^{2t}e^{i(3t-\phi)}}{6\sqrt{29}}, \text{ and } \boxed{x_p(t) = \text{Re}(z_p(t)) = \frac{e^{2t}}{6\sqrt{29}}\cos(3t-\phi).}$$

Again, using the homogeneous solution from Part (a), we have the general solution to the linear DE is $\boxed{x(t) = x_p(t) + x_h(t).}$

(d) $P(D)x = e^{-t}$.

Solution: Since $P(-1) = 0$, we use the extended ERF: $\boxed{x_p(t) = te^{-t}/P'(-1) = te^{-t}/4.}$

Again, using the homogeneous solution from Part (a), we have the general solution to the linear DE is $\boxed{x(t) = x_p(t) + x_h(t).}$

Problem 6.28. (Sinusoidal response formula (SRF))
Let $P(D) = D^2 + 4D + 3$. Find a solution to $P(D)x = \cos(2t)$

Solution: The SRF says a particular solution is given by

$$x_p(t) = \frac{\cos(2t-\phi)}{|P(2i)|}, \text{ where } \phi = \text{Arg}(P(2i)).$$

Computing: $P(2i) = -1 + 8i$, so $|P(2i)| = \sqrt{65}$ and $\boxed{\phi = \text{Arg}(P(2i)) = \tan^{-1}(-8) \text{ in Q2}}$.

Thus, $\boxed{x_p(t) = \dfrac{\cos(2t-\phi)}{\sqrt{65}}}$.

Problem 6.29. *Solve $P(D)x = x'' + 4x' + 5x = e^{-t}\cos 2t$.*

Do this using complex replacement. Give the general solution.

Solution: Particular solution:
Complex replacement:

$$P(D)z = e^{-t}e^{2ti} = e^{(-1+2i)t} \qquad (x = \text{Re}(z))$$

ERF: $z_p(t) = \dfrac{e^{(-1+2i)t}}{P(-1+2i)} = \dfrac{e^{(-1+2i)t}}{-2+4i}$.

Side work: $-2+4i = 2\sqrt{5}e^{i\phi}$, where $\tan\phi = -2$, ϕ in second quadrant.

So, $z_p(t) = \frac{e^{-t}}{2\sqrt{5}}e^{(2t-\phi)i}$. Taking the real part, $x_p(t) = \text{Re}(z_p) = \frac{e^{-t}}{2\sqrt{5}}\cos(2t-\phi)$.

Homogeneous solution: $(P(D)x = 0)$

Characteristic equation: $r^2+4r+5=0 \Rightarrow r = -2 \pm i$.

So the general homogeneous solution is $x_h(h) = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t$.

General solution to $P(D)x = e^{-t}\cos 2t$: $\boxed{x(t) = x_p + x_h = \frac{e^{-t}}{2\sqrt{5}}\cos(2t-\phi) + c_1e^{-2t}\cos t + c_2e^{-2t}\sin t.}$

Problem 6.30. *Let $P(D) = D^2+4D+6I$. Solve $P(D)x = \cos(2t)$.*

Solution: The SRF says a particular solution is given by

$$x_p(t) = \frac{\cos(2t-\phi)}{|P(2i)|}, \quad \text{where } \phi = \text{Arg}(P(2i)).$$

Computing: $P(2i) = 2+8i$, so $|P(2i)| = \sqrt{68}$ and $\boxed{\phi = \text{Arg}(P(2i)) = \tan^{-1}(4) \text{ in Q1}}$.

Thus, $\boxed{x_p(t) = \frac{\cos(2t-\phi)}{\sqrt{68}}}$.

The characteristic equation is $r^2+4r+6=0$. This has roots $r = -2\pm\sqrt{2}\,i$.

General homogeneous solution: $\boxed{x_h(t) = c_1e^{-2t}\cos(\sqrt{2}t) + c_2e^{-2t}\sin(\sqrt{2}t)}$.

General solution to the inhomogeneous DE: $\boxed{x(t) = x_p(t) + x_h(t)}$

Problem 6.31. **(a)** *Solve $x'' + 4x = \cos(\omega t)$ for all possible values of ω.*

Solution: We know that $P(i\omega) = 4-\omega^2$. In polar form we have

$$P(i\omega) = \begin{cases} |4-\omega^2| & \text{if } 0<\omega<2 \\ 0 & \text{if } \omega = 2 \\ |4-\omega^2|e^{i\pi} & \text{if } \omega > 2 \end{cases}$$

So, for $\omega \neq 2$, the SRF gives a particular solution

$$x_p(t) = \begin{cases} \frac{\cos(\omega t)}{|4-\omega^2|} & \text{if } 0<\omega<2 \\ \frac{\cos(\omega t-\pi)}{|4-\omega^2|} = -\frac{\cos(\omega t)}{|4-\omega^2|} & \text{if } \omega>2 \end{cases}$$

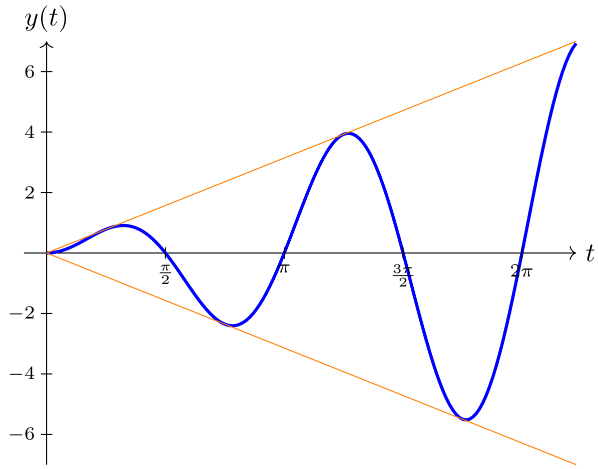
For $\omega = 2$ we need the extended SRF. First we compute $P'(2i) = 4i = 4e^{\pi i/2}$. So,

$$\boxed{x_p(t) = \frac{t\cos(2t-\pi/2)}{4} = \frac{t\sin(2t)}{4}.}$$

The homogeneous solution is $\boxed{x_h(t) = c_1\cos(2t) + c_2\sin(2t).}$ As always, the general solution to the linear DE is $\boxed{x(t) = x_p(t) + x_h(t).}$

(b) *Plot the graph of your particular solution for $\omega = 2$.*

Solution: We have $x_p(t) = t\sin(2t)/4$.



Resonant response

Problem 6.32. **(a)** *Show directly from the definition that $P(D) = D^3 + 6D^2 + 7I$ is a linear operator.*

Solution: We have to apply $P(D)$ to a linear combination of functions and see that it behaves properly, i.e., that for functions x_1, x_2 and constants c_1, c_2

$$P(D)(c_1x_1 + c_2x_2) = c_1P(D)x_1 + c_2P(D)x_2.$$

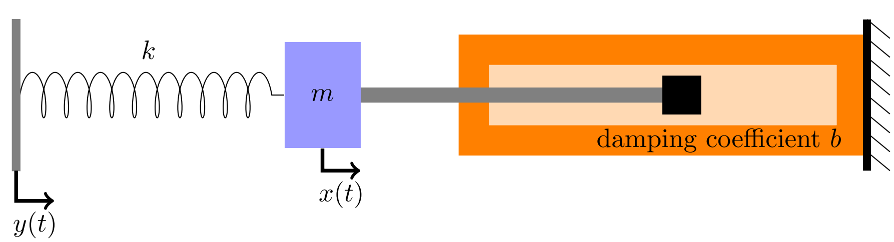
This is always a simple, but tedious, calculation

$$\begin{aligned} P(D)(c_1x_1 + c_2x_2) &= (c_1x_1 + c_2x_2)''' + 6(c_1x_1 + c_2x_2)'' + 7(c_1x_1 + c_2x_2) \\ &= c_1(x_1''' + 6x_1'' + 7x_1) + c_2(x_2''' + 6x_2'' + 7x_2) \\ &= c_1P(D)x_1 + c_2P(D)x_2 \quad \blacksquare \end{aligned}$$

(b) *Say to yourself: "Checking linearity is always easy. You just have to remember to ask."*

Solution: Oh, go on, just say it.

Problem 6.33. *Driving through the spring. Suppose the spring-mass-dashpot is driven by a mechanism that positions the end of the spring at $y(t)$ as shown. As before, $x(t)$ is the position of the mass. We calibrate x and y so that $x = 0$, $y = 0$ is an equilibrium position of the system.*



Give the DE modeling the position $x(t)$ of the mass. Assume, m, k, b, x, y are in compatible units.

Since we control $y(t)$, it is the input. To model $x(t)$, we must consider all the forces on the mass. At time t, the spring is stretched an amount $x(t) - y(t)$, so the spring force is $-k(x - y)$. Likewise, the velocity of the damper through the dashpot is $\dot{x}$. So the damping force is $-b\dot{x}$. Thus, using Newton's second law,

$$m\ddot{x} = -k(x - y) - b\dot{x} \quad \Leftrightarrow \quad m\ddot{x} + b\dot{x} + kx = ky.$$

Topic 7: Undetermined coefficients for polynomial input

Problem 7.34. *(Example from Topic 7 notes.)*
Solve $y'' + 5y' + 4y = 2t + 3$ by the method of undetermined coefficients.

Solution: First, we find a particular solution using the method of undetermined coefficients:

We guess a trial solution of the form $y_p(t) = At + B$. Our guess has the same degree as the input.

Substitute the guess into the DE and do the algebra to compute the coefficients. Here is one way to present the calculation

$$\begin{array}{rcrcl} y_p & = & At & + & B \\ y_p' & = & & & A \\ y_p'' & = & 0 & & \\ y_p'' + 5y_p' + 4y_p & = & 4At & + & (5A + 4B) \end{array}$$

Substituting this into the DE we get:

$$4At + (5A + 4B) = 2t + 3.$$

Now, we equate the coefficients on both sides to get two equations in two unknowns.

$$\begin{array}{llll} \text{Coefficients of } t: & 4A & & = 2 \\ \text{Coefficients of } 1: & 5A & + 4B & = 3 \end{array}$$

This is called a triangular system of equations. First we find $A = 1/2$ and then $B = 1/8$.
So, $\boxed{y_p(t) = \frac{1}{2}t + \frac{1}{8}.}$

Next, we find the solution to the associated homogeneous DE: $y'' + 5y' + 4y = 0$.

Characteristic equation: $r^2 + 5r + 4 = 0 \quad \Rightarrow$ roots are $r = -1, -4$.

General homogeneous solution: $\boxed{y_h(t) = c_1e^{-t} + c_2e^{-4t}.}$

Finally, we use the superposition principle to write the general solution to our DE:

$$y(t) = y_p(t) + y_h(t) = \frac{1}{2}t + \frac{1}{8} + c_1e^{-t} + c_2e^{-4t}.$$

Problem 7.35. *Solve $x' + 3x = t^2 + t$.*

Solution: Guess a trial solution of the form $x_p(t) = At^2 + Bt + C$ (same degree as the input). Substitute the guess into the DE (we don't show the algebra):

$$x_p' + 3x_p = 3At^2 + (2A + 3B)t + (B + 3C) = t^2 + t.$$

Equate the coefficents of the polynonials on both sides of the equation:

$$\begin{array}{lllllll} \text{Coeff. of } t^2\text{:} & 3A & & & & & = 1 \\ \text{Coeff. of } t\text{:} & 2A & + & 3B & & & = 1 \\ \text{Coeff. of } 1\text{:} & & + & B & +3C & & = 0 \end{array}$$

This triangular system is easy to solve: $A = 1/3$, $B = 1/9$, $C = -1/27$. Therefore, a particular solution is

$$\boxed{x_p(t) = \frac{1}{3}t^2 + \frac{1}{9}t - \frac{1}{27}.}$$

The homogeneous solution is $\boxed{x_h(t) = Ce^{-3t}.}$

The general solution to the linear DE is $\boxed{x(t) = x_p(t) + x_h(t).}$

Problem 7.36. *Find a particular solution to $x'' + x' = t^4$.*

Write down the system of equations for A, B, C, D, E, but don't bother solving.

Solution: We try $x_p = At^5 + Bt^4 + Ct^3 + Dt^2 + Et^1$. (Because the lowest derivative in the DE is x' must increase all degrees in the guess by 1.)

Not showing all the algebra, we have

$$x_p'' + x_p' = 5At^4 + (20A + 4B)t^3 + (12B + 3C)t^2 + (6C + 2D)t + (2D + E) = t^4.$$

Equating coefficients we get the system of equations

$$\begin{array}{rlllll} \text{Coeff. of } t^4\text{:} & 5A & & & & = 1 \\ \text{Coeff. of } t^3\text{:} & 20A{+}4B & & & & = 0 \\ \text{Coeff. of } t^2\text{:} & & 12B{+}3C & & & = 0 \\ \text{Coeff. of } t\text{:} & & & 6C{+}2D & & = 0 \\ \text{Coeff. of } 1\text{:} & & & & 2D{+}E & = 0 \end{array}$$

Homogeneous solution: The characteristic equation is $r^2 + r = 0$. This has roots $r = 0, -1$. The general homogeneous solution is $\boxed{x_h(t) = c_1 + c_2e^{-t}.}$

The general solution to the DE is

$$\boxed{x(t) = x_p(t) + x_h(t) = At^5 + Bt^4 + Ct^3 + Dt^2 + Et + c_1 + c_2e^{-t}.}$$

Topic 8: Applications: stability

Problem 8.37. *Is the system $x'' + x' + 4x = 0$ stable?*

Solution: Short answer: second-order with positive coefficients implies $\boxed{\textit{stable}}$.

Longer answer: characteristic roots are $r = \frac{-1\pm\sqrt{1-16}}{2}$. Since both roots have a negative real part, the system is stable.

Problem 8.38. *Is a 4th order system with roots $\pm 1, -2 \pm 3i$ stable. Which solutions to the homogeneous DE go to 0 as $t \to \infty$?*

Solution: No, the root $r = 1$ is positive so the system is not stable.

The general homogeneous solution is

$$x_h(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t}\cos(3t) + c_4 e^{-2t}\sin(3t),$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

The solutions that go to 0 are the ones with $c_1 = 0$, i.e., those of the form

$$x(t) = c_2 e^{-t} + c_3 e^{-2t}\cos(3t) + c_4 e^{-2t}\sin(3t),$$

where c_2, c_3, c_4 are arbitrary constants.

Problem 8.39. *For what k is the system $x' + kx = 0$ stable?*

Solution: Since the characteristic root is $r = -k$, this is stable when $k > 0$.

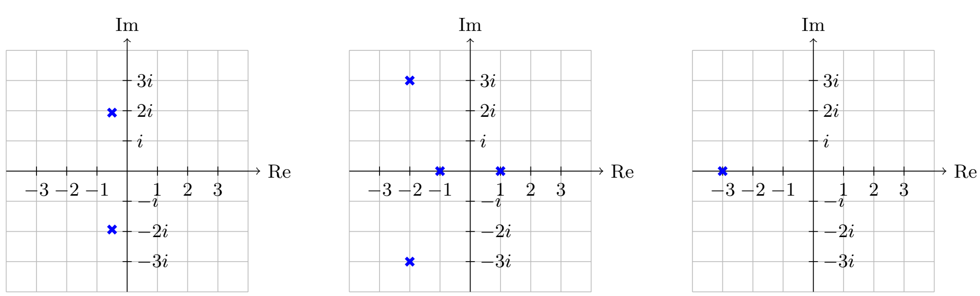
A better way to see this is, if $k > 0$ the system is one of exponential decay. If $k < 0$ it is one of exponential growth. If $k = 0$ it is an edge case. Some people will say it's stable but not asymptotically stable.

Problem 8.40. *Consider the following systems.*

(i) $x'' + x' + 4x = 0$
(ii) A fourth-order system with roots ± 1, $-2 \pm 3i$
(iii) $x' + 3x = 0$.

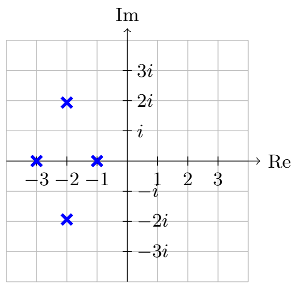
Draw the pole diagram for each of these systems and say how it relates to the stability of the system.

Solution: The pole diagrams are shown in order for (i), (ii) and (iii).

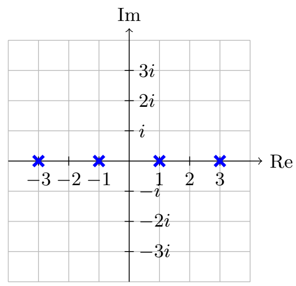


Stability requires all the roots have negative real parts. That is, all the poles are in the left half plane. We see that (i) and (iii) are stable, but (ii) is not.

Problem 8.41. **(a)** *The pole diagram below on the left shows the characteristic roots of the system $P(D)x = 0$.*



Left: pole diagram for Part (a).



Right: diagram for Part (b)

(i) What is the order of the system?

(ii) Is the system stable?

(iii) Is the system oscillatory?

(iv) What is the exponential decay rate for the general solution?

Solution: (i) There are 4 roots, so the order of the system is 4.

(ii) All the roots have negative real part, so the system is stable.

(iii) Since some of the roots are complex, the system is oscillatory.

(iv) The root with the least negative real part, i.e., the right-most root, controls the decay rate. The general solution decays like e^{-t}.

(b) *Repeat Part (a) for the pole diagram on the right.*

Solution: (i) There are 4 roots, so the order of the system is 4.

(ii) Some roots have positive real part, so the system is unstable.

(iii) All the roots are real, so the system is not oscillatory.

(iv) The general system does not decay, it grows like e^{3t}. We could make a case for saying the system decays like e^{3t}.
