

# ES.1803 Problem Section Problems for Quiz 3, Spring 2024 Solutions

## Topic 9: Amplitude response, resonance and practical resonance

**Problem 9.1.** Consider the system  $x'' + 8x = F_0 \cos(\omega t)$ .

(a) Why is this called a driven undamped system?

**Solution:** This models an undamped harmonic oscillator (with mass = 1). The input  $F_0 \cos(\omega t)$  is a force which ‘drives’ the motion of system.

Note: we assumed that  $F_0 \cos(\omega t)$  is the input. Not matter what we call it, the right side of the equation represents a force that drives the system.

(b) Solve this using the sinusoidal response formula (SRF). Then do it again using complex replacement and the exponential response formula (ERF).

**Solution:** Using the SRF and Extended SRF we have

$$\text{For } \omega \neq \sqrt{8}: P(i\omega) = 8 - \omega^2 = |8 - \omega^2|e^{i\phi(\omega)}, \text{ where } \phi(\omega) = \begin{cases} 0 & \text{if } 8 - \omega^2 > 0 \\ \pi & \text{if } 8 - \omega^2 < 0. \end{cases}$$

$$\text{So, } x_p(t) = \frac{F_0 \cos(\omega t - \phi(\omega))}{|8 - \omega^2|} = \begin{cases} F_0 \cos(\omega t) & \text{if } \omega < \sqrt{8} \\ F_0 \cos(\omega t - \pi) = -F_0 \cos(\omega t) & \text{if } \omega > \sqrt{8}. \end{cases}$$

$$\text{If } \omega = \sqrt{8}, \text{ we need the Extended SRF: } x_p(t) = \frac{F_0 t \cos(\omega t - \phi)}{|P'(i\omega)|}, \text{ where } \phi = \text{Arg}(P'(i\omega)).$$

$P'(r) = 2r$ , so  $P'(i\sqrt{8}) = 2i\sqrt{8} = 2\sqrt{8}e^{i\pi/2}$ . Therefore,

$$x_p(t) = \frac{F_0 t \cos(\sqrt{8}t - \pi/2)}{2\sqrt{8}}.$$

Alternatively, we could use complex replacement to arrive at the same formula.

Complexify:  $z'' + 8z = F_0 e^{i\omega t}$ ,  $x = \text{Re}(z)$ .

Characteristic polynomial:  $P(r) = r^2 + 8$ .

Exponential response formula: As above,  $P(i\omega) = |P(i\omega)|e^{i\phi}$ , where

$$|P(i\omega)| = |8 - \omega^2| \quad \text{and} \quad \phi(\omega) = \text{Arg}(P(i\omega)) = \begin{cases} 0 & \text{if } 8 - \omega^2 > 0 \\ \pi & \text{if } 8 - \omega^2 < 0. \end{cases}$$

So, if  $\omega \neq \sqrt{8}$ , then

$$z_p(t) = \frac{F_0 e^{i\omega t}}{P(i\omega)} = \frac{F_0 e^{i(\omega t - \phi(\omega))}}{|8 - \omega^2|}, \text{ so } x_p(t) = \text{Re}(z(t)) = \frac{F_0 \cos(\omega t - \phi(\omega))}{|8 - \omega^2|}$$

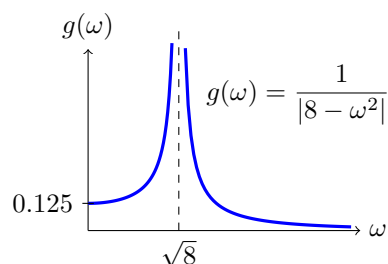
If  $\omega = \sqrt{8}$ , then  $P(i\omega) = 0$ , and we need to use the extended exponential response formula.

As usual, we label the natural frequency  $\omega_0 = \sqrt{8}$ .

$$z_p(t) = \frac{F_0 t e^{i\omega_0 t}}{P'(i\omega_0)} = \frac{F_0 t e^{i\omega_0 t}}{2i\omega_0} = \frac{F_0 t e^{i\omega_0 t}}{2\omega_0 e^{i\pi/2}} = \frac{F_0 t e^{i(\omega_0 t - \pi/2)}}{2\omega_0} \quad \text{so, } x(t) = \text{Re}(z(t)) = \frac{F_0 t \cos(\omega_0 t - \pi/2)}{2\omega_0}.$$

(c) Consider the right hand side of the DE to be the input. Graph the amplitude response function.

**Solution:**  $g(\omega) = 1/|8 - \omega^2|$ . The graph of  $g$  vs.  $\omega$  has a vertical asymptote at  $\omega = \sqrt{8}$ .



(d) What is the resonant frequency of the system?

**Solution:** Resonant frequency at  $\omega = \sqrt{8}$ .

(e) Why is this called the natural frequency?

**Solution:** Because the unforced system  $x'' + 8x = 0$  will oscillate at this frequency

**Problem 9.2.** Consider the forced damped system:  $x'' + 2x' + 9x = \cos(\omega t)$ .

(a) What is the natural frequency of the system?

**Solution:** Natural frequency = frequency of unforced, undamped system =  $\sqrt{9} = 3$ .

(b) Find the response of the system in amplitude-phase form.

**Solution:** We can go right to the sinusoidal response formula:  $x_p(t) = \frac{1}{|P(i\omega)|} \cos(\omega t - \phi(\omega))$ ,

where,  $P(r) = r^2 + 2r + 9 \Rightarrow P(i\omega) = 9 - \omega^2 + 2i\omega$ .

So,  $|P(i\omega)| = \sqrt{(9 - \omega^2)^2 + 4\omega^2}$ , and  $\phi(\omega) = \text{Arg}(P(i\omega)) = \tan^{-1} \frac{2\omega}{9 - \omega^2}$  in Q1 or Q2.

Alternatively (and you should know how to do this)

Complexify:  $z'' + 2z' + 9z = e^{i\omega t}$ ,  $x = \text{Re}(z)$ .

Char. polynomial:  $P(r) = r^2 + 2r + 9$ , so  $P(i\omega) = 9 - \omega^2 + 2i\omega$ .

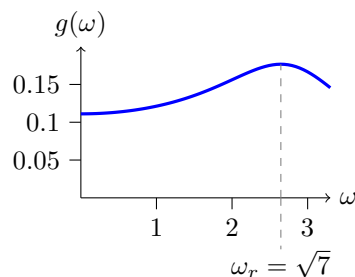
ERF:  $z_p(t) = \frac{e^{i\omega t}}{P(i\omega)} = \frac{e^{i\omega t}}{9 - \omega^2 + 2i\omega}$

Polar form:  $9 - \omega^2 + 2i\omega = \sqrt{(9 - \omega^2)^2 + 4\omega^2} e^{i\phi(\omega)}$ , where  $\phi(\omega) = \tan^{-1}(2\omega/(9 - \omega^2))$ , in Q1 or Q2.

Thus,  $z_p(t) = \frac{1}{\sqrt{(9 - \omega^2)^2 + 4\omega^2}} e^{i(\omega t - \phi(\omega))}$ . This implies  $x_p(t) = \frac{1}{\sqrt{(9 - \omega^2)^2 + 4\omega^2}} \cos(\omega t - \phi(\omega))$ .

(c) Consider the right hand side of the DE to be the input. What is the amplitude response of the system? Draw its graph –be sure to label your axes correctly

**Solution:** Amplitude response = gain =  $g(\omega) = \frac{1}{\sqrt{(9 - \omega^2)^2 + 4\omega^2}}$ .



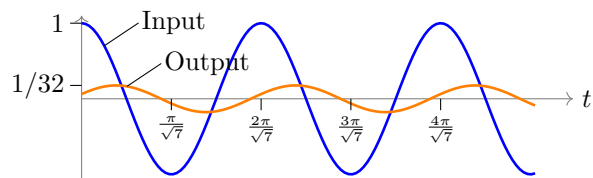
Gain curve and practical resonant frequency

(d) *What is the practical resonant frequency?*

**Solution:** Practical resonance is where  $g(\omega)$  has a maximum. This is the same as where the expression under the radical,  $(9 - \omega^2)^2 + 4\omega^2$ , has a minimum. Simple calculus shows this is at  $\omega_r = \sqrt{7}$ .

(e) *When  $\omega = \sqrt{7}$  by how many radians does the output peak lag behind the input peak?*

**Solution:** If  $\omega = \sqrt{7}$  then  $\phi = \tan^{-1} \sqrt{7} = 1.2$  radians and  $g(\sqrt{7}) = 1/\sqrt{32} \approx 0.177$ .



**Problem 9.3.** *Consider the system*

$$2y'' + 10y' + 3y = 3B \cos(\omega t),$$

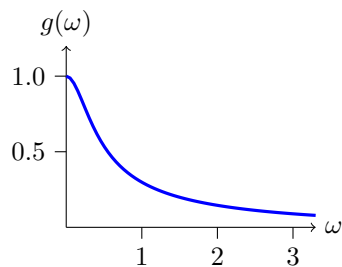
*where we consider  $B \cos(\omega t)$  to be the input. Find and graph the gain. Find the practical resonant frequency.*

**Solution:** The sinusoidal response formula tells us the sinusoidal solution to this equation is

$$y_p(t) = \frac{3B \cos(\omega t - \phi(\omega))}{|P(i\omega)|}, \quad \text{where } \phi(\omega) = \text{Arg}(P(i\omega)).$$

Since the input is  $B \cos(\omega t)$ , the gain is

$$g(\omega) = \frac{3}{|P(i\omega)|} = \frac{3}{\sqrt{(3 - 2\omega^2)^2 + 100\omega^2}}$$



Finding the practical resonant frequency means finding the value of  $\omega$  that maximizes the gain:

$$g'(\omega) = -\frac{3}{2} \cdot \frac{-8\omega(3 - 2\omega^2) + 200\omega}{((3 - 2\omega^2)^2 + (10\omega)^2)^{3/2}} = 0.$$

Setting the numerator to 0 and solving for  $\omega$  we find  $\omega = 0$  or  $\omega = \sqrt{-11}$ . Since neither of these is a positive real number we say that there is **no practical resonant frequency**.

**Note:** Just to change things up, we computed  $g'(\omega)$ . We could have used our usual trick and found the minimum of  $h(\omega) = \frac{1}{g(\omega)^2} = (3 - 2\omega^2)^2 + 100\omega^2$ . Of course,  $h'(\omega)$  is just the numerator of  $g'(\omega)$ .

**Problem 9.4.** *For the forced undamped system  $x'' + 8x = F_0 \cos(\omega t)$ , give a detailed description of the phase lag for different input frequencies. (Consider  $F_0 \cos(\omega t)$  to be the input.)*

**Solution:** Because  $P(i\omega) = 8 - \omega^2$  is real, we have

$$\phi(\omega) = \begin{cases} 0 & \text{if } 8 - \omega^2 < 0, \text{ i.e., } \omega < \omega_0 \\ \pi & \text{if } 8 - \omega^2 > 0, \text{ i.e., } \omega > \omega_0. \end{cases}$$

When  $\omega = \omega_0$  there is no periodic solution so officially there is no phase lag but, looking at the solution  $x_p(t) = t \cos(\omega t - \pi/2)/2\omega_0$ , it seems that  $\phi = \pi/2$  is a reasonable choice.

**Problem 9.5.** *Consider the driven first-order system:  $x' + kx = kF_0 \cos(\omega t)$ . We'll take the input to be  $F_0 \cos(\omega t)$ . Solve the DE. Find the amplitude response. Show there is never practical resonance.*

**Solution:** Use the sinusoidal response formula:  $x_p(t) = \frac{kF_0 \cos(\omega t - \phi(\omega))}{|P(i\omega)|}$ .

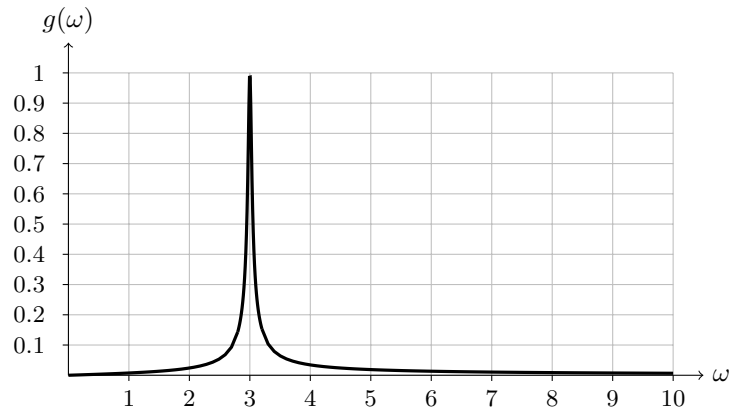
Characteristic polynomial:  $P(r) = r + k$ . So,  $P(i\omega) = i\omega + k$ .

Polar form:  $P(i\omega) = k + i\omega \Rightarrow |P(i\omega)| = \sqrt{k^2 + \omega^2}$ , and  $\phi(\omega) = \text{Arg}(P(i\omega)) = \tan^{-1}(\omega/k)$ , in Q1.  
So,

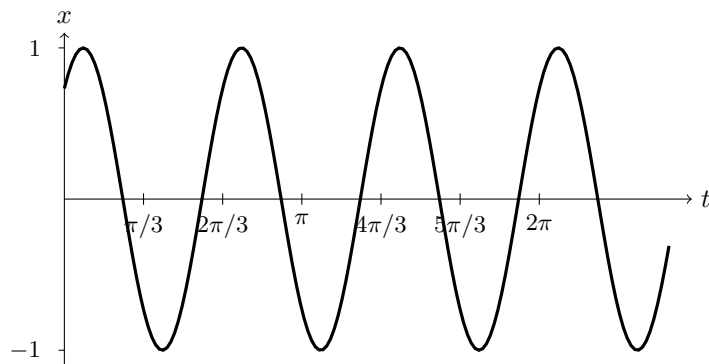
$$x_p(t) = \frac{kF_0}{\sqrt{k^2 + \omega^2}} \cos(\omega t - \phi(\omega))$$

Amplitude response = gain =  $g(\omega) = \frac{k}{\sqrt{k^2 + \omega^2}}$ . This is a decreasing function in  $\omega$ , so there is no positive local maximum, i.e., no practical resonance.

**Problem 9.6.** *Below is a gain curve. Suppose the input is  $\sum_{n=0}^{100} \cos(nt)$ . Give a rough sketch of the output.*



**Solution:** The gain curve will be 1 near  $n = 3$  and near 0 for all other  $n$ . Therefore, the system will (essentially) filter out all the frequencies except  $n = 3$ . The output will look very much like  $x = \cos(3t)$ .



(In reality, you would see a little wobble due to the small contribution of  $n$  near 3.)

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