# ES.1803 Quiz 4 Solutions, Spring 2024

**Problem 1.** (20 points)

*Let* $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 2 & 3 & 7 \\ 2 & 4 & 6 & 14 \end{bmatrix}$

**(a)** (10) *Put $A$ in reduced row echelon form.*

**Solution:** $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 2 & 3 & 7 \\ 2 & 4 & 6 & 14 \end{bmatrix} \xrightarrow[\text{Row}_3\text{-2Row}_1]{\text{Row}_2\text{-Row}_1} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix} \xrightarrow{\text{Row}_3\text{-2Row}_2} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row}_2/2}$

$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row}_1\text{-Row}_2} \boxed{\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.}$

**(b)** (5) *Give a basis for the column space of $A$.*

**Solution:** The pivot columns are Columns 1 and 3. These columns of $A$ give a basis for the column space: $\boxed{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.}$

**(c)** (3) *What is the dimension of Null(A)?*

**Solution:** Dimension of Null($A$) = number of free variables = 2.

**(d)** (2) *What is the rank of $A$?*

**Solution:** There are two pivots, so the rank is 2.

**Problem 2.** (20 points)

*The matrix $R$ is in reduced row echelon form:* $R = \begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

**(a)** (10) *Give the general solution to the equation* $R\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}.$

**Solution:** We need Null($R$), i.e., the general homogeneous solution. As usual, we format the solution by putting the variables below the matrix. The free variables are $x_2$ and $x_4$.

In turn, we set one to 1 and the other to 0 and solve for the pivot variables.

$$\begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ 3 & 1 & 0 & 0 \\ -3 & 0 & -2 & 1 \end{matrix}$$

So a basis of Null$(R)$ is $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Since $R$ is in RREF, it is easy to see that $\begin{bmatrix} 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} = 3 \cdot \text{Col}_1 + 4 \cdot \text{Col}_3$. So a particular solution is $\mathbf{x_p} = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix}$. The general solution is $\boxed{\mathbf{x} = \mathbf{x_p} + \mathbf{x_h} = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}}$.

**(b)** (10) *Find a matrix $A$ with reduced row echelon form $R$ and such that the equations $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix}$ and $A\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ can both be solved.*

**Solution:** We put $\begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ as pivot columns of $A$ and give the free columns the same relations to the pivot columns as seen in $R$.

That is $\text{Col}_2 = -3\text{Col}_1$ and $\text{Col}_4 = 3{\cdot}\text{Col}_1 + 2{\cdot}\text{Col}_3$: $\boxed{A = \begin{bmatrix} 2 & -6 & 1 & 8 \\ 1 & -3 & -1 & 1 \\ 4 & -12 & 0 & 12 \\ 1 & -3 & 0 & 3 \end{bmatrix}}$

**Problem 3.** (30 points)

**(a)** (10) *Let $A = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$. Find the general* real-valued *solution to $\mathbf{x}' = A\mathbf{x}$.*

*Helpful check: Your eigenvalues should be integers.*

**Solution:** Characteristic equation: $\det \begin{bmatrix} 6-\lambda & -2 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - 7\lambda + 10 = 0$. So, $\lambda = 2, 5$.

Basic eigenvectors: Need basis of Null$(A - \lambda I)$

For $\lambda = 2$: $A - 2I = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

General (real-valued) solution: $\boxed{\mathbf{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.}$

Note: There are many possible ways to give the answer depending on how you chose your eigenvectors.

**(b)** (10) *Suppose $B$ is a $3 \times 3$ matrix with eigenvalues 3, 7, 10 and corresponding eigenvectors* $\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

*(i) Find* $\det(B)$. *(ii) Find* $B\left(2\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$. *(iii) Find* $B^{-1}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$.

*(iv) Give the general solution to the system of DEs* $\mathbf{x}' = B^2\mathbf{x}$.

**Solution:** (i) $\det(B)$ is the product of the eigenvalues $= 210$.

(ii) Since the vector being multiplied is a linear combination of eigenvectors, the result is

$$B\left(2\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = 6\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 40\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 64 \\ 46 \end{bmatrix}.$$

(iii) Since $\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 3 we have $B^{-1}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$.

(iv) The eigenvalues of $B^2$ are $3^2$, $7^2$, $10^2$. The eigenvectors are the same as for $B$. So the general solution is $\boxed{\mathbf{x} = c_1 e^{9t}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + c_2 e^{49t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{100t}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.}$

**(c)** (5) *Give the diagonalized form of the matrix $B$ from Part (b). That is, write it as a product of certain matrices. You do not have to find inverses explicitly.*

**Solution:** We know $B = S\Lambda S^{-1}$, where $S = \begin{bmatrix} 1 & 1 & 0 \\ 4 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is the matrix of eigenvectors and $\Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 10 \end{bmatrix}$ is the diagonal matrix of eigenvalues. So,

$$B = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 4 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 10 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 4 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

**(d)** (5) *Suppose $D$ is a $2 \times 2$ matrix with eigenvalues $1 + 2i$ and $1 - 2i$ and corresponding eigenvectors $\begin{bmatrix} 1 \\ 3i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3i \end{bmatrix}$.*

*Give the general real-valued solution to the equation $\mathbf{x}' = D\mathbf{x}$.*

**Solution:** One complex solution is $\mathbf{z}(t) = e^{(1+2i)t} \begin{bmatrix} 1 \\ 3i \end{bmatrix}$. So,

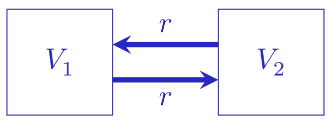
$$\mathbf{z}(t) = e^t(\cos(2t) + i\sin(2t)) \begin{bmatrix} 1 \\ 3i \end{bmatrix} = e^t \begin{bmatrix} \cos(2t) \\ -3\sin(2t) \end{bmatrix} + ie^t \begin{bmatrix} \sin(2t) \\ 3\cos(2t) \end{bmatrix}$$

The real and imaginary parts of $\mathbf{z}$ are both solutions to the DE. So,

$$\mathbf{x_1}(t) = \mathrm{Re}(\mathbf{z}) = e^t \begin{bmatrix} \cos(2t) \\ -3\sin(2t) \end{bmatrix} \quad \text{and} \quad \mathbf{x_2}(t) = \mathrm{Im}(\mathbf{z}) = e^t \begin{bmatrix} \sin(2t) \\ 3\cos(2t) \end{bmatrix}$$

are solutions. The general real-valued solution is $\boxed{\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2}.}$

**Problem 4.** (10 points)
*Consider the closed two-compartment mixing tank system shown. Let $x$, $y$ be the amount of salt in tanks 1, 2 respectively. The volumes $V_1$, $V_2$ and the flow rate $r$ are (positive) constants.*




*Assume compatible units and write down in matrix form the system of DEs governing the amount of salt in the tanks.*

**Solution:**

$$\begin{aligned} x' &= \text{ rate in} - \text{rate out } = r \cdot \frac{y}{V_2} - r \cdot \frac{x}{V_1} \\ x' &= \text{ rate in} - \text{rate out } = r \cdot \frac{x}{V_1} - r \cdot \frac{y}{V_2} \end{aligned}$$

In matrix form this is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{r}{V_1} & \frac{r}{V_2} \\ \frac{r}{V_1} & -\frac{r}{V_2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

**Problem 5.** (20 points)
*Give a short explanation for each answer.*

**(a)** (4) *Suppose $A$ is a square matrix with RREF $R$. True or false: $A$ and $R$ have the same eigenvalues.*

**Solution:** $\boxed{\text{False.}}$ For example, $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$ has eigenvalues 1, 7. Its RREF is $I$, which has eigenvalues 1, 1.

**(b)** (4) *Find the companion system to the DE $x'' + 2x' + 7x = 0$. Give your answer in matrix form.*

**Solution:** Let $y = x'$, The differential equation becomes

$$y' + 2y + 7x = 0 \qquad \Rightarrow y' = -7x - 2y.$$

The companion system is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -7 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

**(c)** (4) *Consider the set of all series of the form* $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$. *Here,* $a_0$, $a_n$, $b_n$ *are parameters that can take arbitrary values. Is this set a vector space?*

**Solution:** Yes. Clearly this set is closed under addition and scalar multiplication.

**(d)** (4) **True or false**: *Suppose* $A$ *is a square matrix. If the linear system* $A\mathbf{x} = \mathbf{0}$ *has more than one solution, then* $\det A = 0$.

**Solution:** $\boxed{\text{True.}}$ More than one solution means the null space of $A$ is nontrivial. So $A$ is not invertible, which implies $\det(A) = 0$.

**(e)** (4) *Suppose* $E$ *is a* $2 \times 2$ *matrix with eigenvalues 1 and -3 and corresponding eigenvectors* $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

*Suppose* $\begin{bmatrix} x \\ y \end{bmatrix}$ *is a solution to the system* $\mathbf{x}' = E\mathbf{x}$. *As* $t$ *gets large, the ratio of* $x$ *to* $y$ *goes asymptotically to what value?*

**Solution:** $\boxed{3/5.}$ $\mathbf{x} = c_1 e^t \begin{bmatrix} 3 \\ 5 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 7 \\ 2 \end{bmatrix}$. As $t$ gets large the second term goes to 0, so we have $x \approx 3c_1 e^t$ and $y \approx 5c_1 e^t$. Therefore, the ratio $x/y$ becomes 3/5.

*End of quiz solutions*