ES.1803 Problem Section Problems for Quiz 4, Spring 2024

Topic 13: Linearity, matrix multiplication, systems of equations, DEs.

Problem 13.1. Compute the following by thinking of matrix multiplication as a linear combination of the columns of the matrix.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$

Problem 13.2. Make up a block matrix problem: Multiply a 4×4 matrix made up of four 2×2 blocks (two blocks of 0s, one block = identity, one block something else) times a 4×2 matrix with (i.e., two 2×2 blocks)

Problem 13.3. Is it a vector space? For all of these you just have to check that they are closed under addition and scalar multiplication, i.e. closed under linear combinations.

(a) The set of functions $f(x)$ such that $f(5) = 0$.

(b) The set of functions $f(x)$ such that $f(5) = 2$.

(c) The set of vectors (x, y) in the plane, such that $2x + 3y = 0$.

(d) The set of vectors (x, y) in the plane, such that $2x + 3y = 2$.

Problem 13.4. Convert the following ODE to a companion system: $x''' + 2x'' + 3x' + 4x = \cos(5t)$.

Topic 14: Linear algebra: row reduction and subspaces

Problem 14.5. Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 4 & 6 & 2 & 4 \\ 3 & 6 & 10 & 3 & 6 \end{bmatrix}$. Put A in row reduced echelon form. Find the rank, a basis of the column space, a basis of the null space, and the dimension of each of the spaces.

Problem 14.6. Let $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Suppose R is the row reduced echelon form for A.

(a) What is the rank of A?

(b) Find a basis for the null space of A.

(c) Suppose the column space of A has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. Find a possible matrix for A. That is, give a matrix A with RREF R and the given column space.

(d) Find a matrix with the same row reduced echelon form, but such that $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\\3\end{bmatrix}$ are in its column space.

Problem 14.7. **(a)** Suppose we have a matrix equation

$$\begin{bmatrix}\bullet\\\bullet\end{bmatrix}\begin{bmatrix}\bullet & \bullet\end{bmatrix} = \begin{bmatrix}1 & 1\\2 & x\end{bmatrix}$$

Can you specify x? For any value of x you think is allowable, find such an equation. Can any of the $\bullet$'s be 0?

(b) Suppose we have a matrix equation

$$\begin{bmatrix}\bullet & 3\\\bullet & 4\\\bullet & 5\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Can you specify the $\bullet$'s?

(c) Suppose we have a matrix equation

$$\begin{bmatrix}x & 3\\y & 4\\z & 5\end{bmatrix}\mathbf{c} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

and all we know about the vector $\mathbf{c}$ is that $\mathbf{c} \neq \mathbf{0}$. What can we say about $\begin{bmatrix}x\\y\\z\end{bmatrix}$?

Problem 14.8. Suppose we have a matrix equation

$$\begin{bmatrix}1 & x & 2\\3 & y & 4\\5 & z & 6\end{bmatrix}\mathbf{c} = \mathbf{0}$$

and all we know about the vector $\mathbf{c}$ is that $\mathbf{c} \neq \mathbf{0}$. What can we say about $\begin{bmatrix}x\\y\\z\end{bmatrix}$?

Problem 14.9. For what values of y is it the case that the columns of $\begin{bmatrix}1 & 1 & 2\\3 & y & 4\\5 & 1 & 6\end{bmatrix}$ form a linearly independent set?

Problem 14.10. For the matrix $A = \begin{bmatrix}1 & 2 & 3\\2 & 3 & 4\\3 & 4 & 5\end{bmatrix}$:

(a) Find the row reduced echelon form of A; call it R.

(b) The last column of R should be a linear combination of the first columns in an obvious way. This is a linear relation among the columns of R. Find a vector $\mathbf{x}$, such that $R\mathbf{x} = \mathbf{0}$, which expresses this linear relationship.

(c) Verify that the same relationship holds among the columns of A.

(d) Explain why the linear relations among the columns af R are the same as the linear relations among the columns of A. In fact, explain why, if A and B are related by row transformations, the linear relations among the columns of A are the same as the linear relations among the columns of B.

Problem 14.11. Suppose we want to solve $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

(a) When is this possible? Answer this in the form: "$\mathbf{b}$ must be a linear combination of the two vectors …"

(b) $A\mathbf{x} = \mathbf{b}$ is certainly solvable for $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. (What is the obvious particular solution?) Describe the general solution to this equation, as $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$.

Problem 14.12. Suppose that the row reduced echelon form of the 4×6 matrix B is

$$R = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a linearly independent set of vectors of which every vector in the null space of B is a linear combination.

(b) Write the columns of B as $\mathbf{b_1}, \mathbf{b_2}, \dots, \mathbf{b_6}$. What is $\mathbf{b_1}$? What can we say about $\mathbf{b_2}$? Which of these vectors are linearly independent of the preceding ones? Express the ones which are not independent as explicit linear combinations of the previous ones. Describe a linearly independent set of vectors of which every vector in the column space of B is a linear combination.

Problem 14.13. Solve this system of linear equations. How many methods can you think of to solve this system?

$$\begin{aligned} x + y &= 5 \\ 3x + 2y &= 7 \end{aligned}$$

Problem 14.14. Consider the following system of equations:

$$\begin{aligned} x + y + z &= 5 \\ x + 2y + 3z &= 7 \\ x + 3y + 6z &= 11 \end{aligned}$$

(a) Write this system of equations as a matrix equation.

(b) Use row reduction to get to row echelon form. What is the solution set?

Problem 14.15. Solve the following equation using row reduction:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) At the end of the row-reduction process, was the last column pivotal or free? Is this related to the absence of solutions?

(b) Find a new vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ such that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has a solution.

Problem 14.16. Show that the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ corresponds to counter-clockwise rotation about the origin by 90 degrees, by computing the effect of this matrix on the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and drawing $\mathbf{v}_1$, $\mathbf{v}_2$, $A\mathbf{v}_1$, $A\mathbf{v}_2$ on the plane.

Topic 15: Transpose, inverse, determinant

Problem 15.17. **(a)** Use row reduction to find the inverse of the matrix $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$.

(b) Use the record of the row operations to compute the determinant of A

Problem 15.18. Use row reduction to find inverses of the following matrices. As you do this, record the row operations carefully for later problems.

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -6 & 2 & -2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 2 & 2 \\ 3 & 5 & 7 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$

(d) $D = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 8 \\ 3 & 2 & 5 \end{bmatrix}$

Problem 15.19. Using just the record of the row operations in Problem 15.18 compute the determinant of each matrix.

Problem 15.20. Compute the transpose of the following matrices.

$A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 6 & 7 \end{bmatrix}. \quad D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

Problem 15.21. Let $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

Show by direct computation that $(AD)^T = (D^T A^T)$.

Problem 15.22. **(a)** Recall the notation for inner product: $\langle \mathbf{v}, \mathbf{w} \rangle$. Assume $\mathbf{v}$ and $\mathbf{w}$ are column vectors. Write the formula for inner product in terms of transpose and matrix multiplication.

(b) Using this definition show $\langle A\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, A^T \mathbf{w} \rangle$.

Topic 16: eigenvalues, diagonalization, decoupling

Problem 16.23. Suppose the 2×2 matrix A has eigenvectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ with eigenvalues 2 and 4 respectively.

(a) Find $A(\mathbf{v_1} + \mathbf{v_2})$.

(b) Find $A(\mathbf{5v_1} + \mathbf{6v_2})$.

(c) Find $A \begin{bmatrix} 4 \\ 9 \end{bmatrix}$

Problem 16.24. **(a)** Without calculation, find the eigenvalues and and basic eigenvectors for $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

(b) Without calculation, find at least one eigenvector and eigenvalue for $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

Problem 16.25. **(b)** Find the eigenvalues and basic eigenvectors of $A = \begin{bmatrix} -3 & 4 \\ 2 & -5 \end{bmatrix}$.

Problem 16.26. **(a)** Find the eigenvalues and basic eigenvectors of $A = \begin{bmatrix} 3 & 1 & -3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$.

(b) Write A in diagonalized form.

(c) Compute A^5.

Problem 16.27. Suppose that the matrix B has eigenvalues 1 and 7, with eigenvectors

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

respectively.

(a) What is the solution to $\mathbf{x}' = Bx$ with $x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$?

(b) Decouple the system $\mathbf{x}' = B\mathbf{x}$. That is, make a change of variables so that system is decoupled. Write the DE in the new variables.

(c) Give an argument based on transformations why $B = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1}$ has the eigenvalues and eigenvectors given above.

Problem 16.28. Suppose $A = \begin{bmatrix} a & b & c \\ 0 & 2 & e \\ 0 & 0 & 3 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) For what value (or values) of a, b, c, e is A singular (non-invertible)?

(c) What is the minimum rank of A (as a, b, c, e vary)? What's the maximum?

(d) Suppose $a = -5$. In the system $\mathbf{x}' = A\mathbf{x}$, is the equilibrium at the origin stable or unstable.

Problem 16.29. Suppose that $A = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} S^{-1}$.

(a) What are the eigenvalues of A?

(b) Express A^2 and A^{-1} in terms of S.

(c) What would I need to know about S in order to write down the most rapidly growing exponential solution to $\mathbf{x}' = A\mathbf{x}$?

Problem 16.30.

(a) An orthogonal matrix is one where the columns are orthonormal (mutually orthogonal and unit length). Equivalently, S is orthogonal if $S^{-1} = S^T$.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Find an orthogonal matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$

(b) Decouple the equation $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Problem 16.31. Find the eigenvalues and basic eigenvectors of $A = \begin{bmatrix} -3 & 13 \\ -2 & -1 \end{bmatrix}$.

Topic 17: Matrix methods of solving systems of DEs

Problem 17.32. **(a)** Let $A = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}$. Solve $\mathbf{x}' = A\mathbf{x}$.

(b) What is the solution to $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

(c) Decouple the system in Part (a). That is, make a change of variables that converts the system to a decoupled one. Write the system in the new variables.

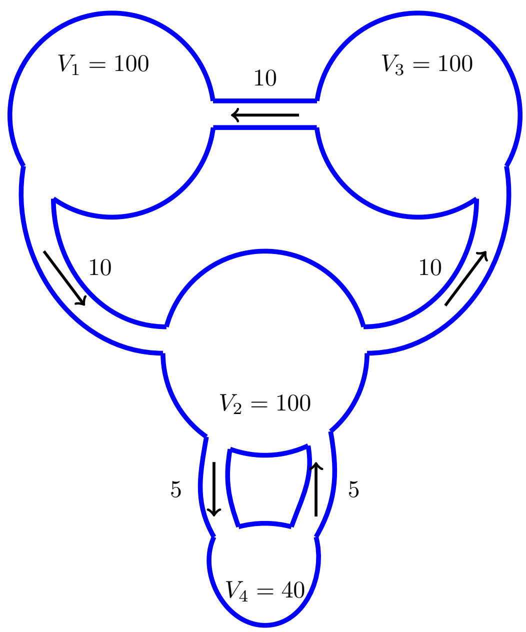
Problem 17.33. Solve $x' = -3x + y, \quad y' = 2x - 2y$.

Problem 17.34. (**Complex roots**) Solve $\mathbf{x}' = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix} \mathbf{x}$ for the general real-valued solution.

Problem 17.35. (**Repeated roots**) Solve $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}$.

Problem 17.36. Solve the system $x' = x + 2y; \quad y' = -2x + y$.

Problem 17.37. The following figure shows a closed tank system with volumes and flows as indicated (in compatible units). Let's call the tank with $V_1 = 100$ tank 1, etc.



(a) Write down a system of differential equations modeling the amount of solute in each tank.

(b) Without computation you know one eigenvalue. What is it? What is a corresponding eigenvector?

(c) What can you say about all the other eigenvalues?