ES.1803 Quiz 5 Solutions, Spring 2024

Problem 1. (40 points)

(a) (5) *Compute* $\displaystyle\int_{0^-}^{8^-} t^2\Big(\delta(t)+2\delta(t-4)+4\delta(t-8)+6\delta(t-16)\Big)\,dt.$

Solution: The spikes at $t=0,4$ are in the interval of integration and those at $t=8,16$ are not. So the integral equals $\boxed{0^2+2\cdot 4^2=32}$.

(b) (5) *Compute* $\displaystyle\int e^{2t}\Big(\delta(t)+2\delta(t-3)\Big)\,dt.$

Solution: We make use of the fact that $f(t)\delta(t-a)=f(a)\delta(t-a)$. So,

$$e^{2t}\Big(\delta(t)+2\delta(t-3)\Big)=\delta(t)+2e^6\delta(t-3).$$

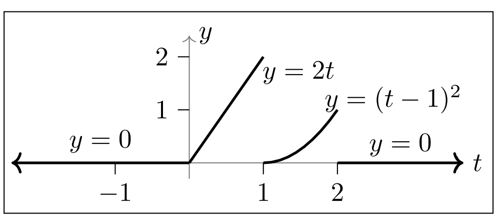
The indefinite integral is $\boxed{u(t)+2e^6u(t-3)+c.}$

(c) (5) *Consider the following function*

$$f(t)=\begin{cases}0 & \text{for } t<0\\ 2t & \text{for } 0<t<1\\ (t-1)^2 & \text{for } 1<t<2\\ 0 & \text{for } 2<t\end{cases}$$

Find the generalized derivative of $f(t)$.

Solution: Looking at the graph of $y=f(t)$, we have to take the generalized derivative to handle the jumps.



$$f'(t)=\underbrace{\begin{cases}0 & t<0\\ 2 & 0<t<1\\ 2(t-1) & 1<t<2\\ 0 & 2<t\end{cases}}_{\text{regular part}}\ \underbrace{-\,2\delta(t-1)-\delta(t-2)}_{\text{singular part}}$$

(d) (15) *Find the solution to the following DE with the given initial conditions.*

$$2x''+8x=3\delta(t-\pi);\quad x(0)=0,\ x'(0)=2.$$

Solution: We'll break this into cases before and after the impulse

Case $t<\pi$: DE: $2x''+8x=0$; IC: $x(0)=0,\ \ x'(0)=2.$

General solution: $x(t)=c_1\cos(2t)+c_2\sin(2t)$.

IC: $x(0)=c_1=0.\quad x'(0)=2c_2=2\quad\longrightarrow c_2=1.$

So, $x(t)=\sin(2t)$.

In preparation for the next case: $x(\pi^-)=0,\ \ x'(\pi^-)=2.$

Case $t < \pi$: DE: $2x'' + 8x = 0$; IC: $x(\pi^+) = x(\pi^-) = 0$, $x'(\pi^+) = x'(\pi^-) + 3/2 = 7/2$.

General solution: $x(t) = c_1 \cos(2t) + c_2 \sin(2t)$.

IC: $x(\pi^+) = c_1 = 0$. $x'(\pi^+) = 2c_2 = 7/2 \quad \longrightarrow c_2 = 7/4$.

So, $x(t) = \frac{7}{4}\sin(2t)$.

All together: $\boxed{x(t) = \begin{cases} \sin(2t) & \text{for } t < \pi \\ \frac{7}{4}\sin(2t) & \text{for } \pi < t \end{cases}}$

(e) (10) *Let $g(t)$ be the period 2 function*

$$g(t) = ... + 2\delta(t+4) + 2\delta(t+2) + 2\delta(t) + 2\delta(t-2) + 2\delta(t-4) + 2\delta(t-6) + ...$$

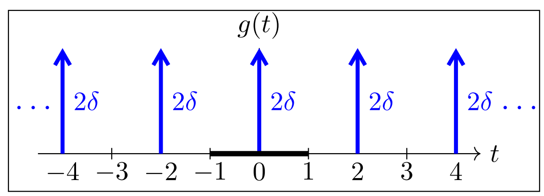
Find the Fourier series of $g(t)$.

Solution: The half-period is $L = 1$. We integrate over one period from -1 to 1. The only $g(t)$ term in this interval is $2\delta(t)$. So,

$$a_n = \frac{1}{1}\int_{-1}^{1} 2\delta(t)\cos(n\pi t)\,dt = 2\cos(0) = 2, \qquad a_0 = \frac{1}{1}\int_{-1}^{1} g(t)\,dt = \int_{-1}^{1} 2\delta(t)\,dt = 2.$$

$$b_n = \frac{1}{1}\int_{-1}^{1} 2\delta(t)\sin(n\pi t)\,dt = 2\sin(0) = 0$$

So, $g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2n\pi t) = \boxed{1 + \sum_{n=1}^{\infty} 2\cos(n\pi t)}$.



Note: Since $g(t)$ is even, we knew without integration that $b_n = 0$.

Problem 2. (15 points)

The periodic function $f(t)$ has Fourier series $f(t) = 3 + 5\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2+1}$. Find the periodic solution to the DE $3x' + x = f(t)$.

Solution: We solve one term at a time and then use superpostion.

Constant term: $3x_0' + x_0 = 3 \Rightarrow x_{0,p}(t) = 3$.

Other terms: $3x_n' + x_n = \cos(nt)$. Using the SRF:

$P(in) = 1{+}3in$. Thus, $|P(in)| = \sqrt{1+9n^2}$, and $\boxed{\phi(n) = \arg(P(in)) = \tan^{-1}\left(\frac{3n}{1}\right) \text{ in Q1.}}$

So, $x_{n,p}(t) = \frac{\cos(nt - \phi(n))}{|P(in)|} = \frac{\cos(nt-\phi(n))}{\sqrt{1+9n^2}}$.

So, by superposition, $\boxed{x_p(t) = 3 + 5\sum_{n=1}^{\infty} \frac{\cos(nt - \phi(n))}{(n^2+1)\sqrt{1+9n^2}}}$.

Problem 3. (10 points)
Find the Fourier cosine series for $f(x) = 1 + x$ on the interval $[0, \pi]$.

Solution: The cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$, where, for $n \neq 0$:

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} (1+x)\cos(nx)\,dx \\ &= \frac{2}{\pi}\left[\int_0^{\pi} \cos(nx)\,dx + \int_0^{\pi} x\cos(nx)\,dx\right] \\ &= \frac{2}{\pi}\left[0 + \begin{cases} -2/n^2 & n \text{ odd} \\ 0 & n \neq 0 \text{ even} \end{cases}\right] = \begin{cases} -\frac{4}{\pi n^2} & n \text{ odd} \\ 0 & n \neq 0 \text{ even} \end{cases} \end{aligned}$$

(The second integral was computed using the integral table.) For a_0:

$a_0 = \frac{2}{\pi}\int_0^{\pi} (1+x)\,dx = 2 + \pi$. So, $\boxed{f(x) = 1 + \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(nx)}{n^2}}$.

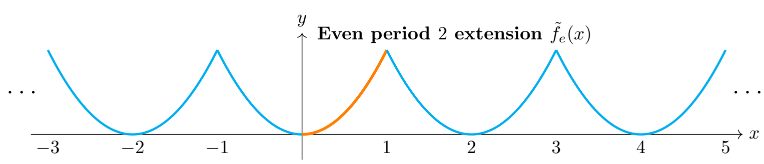
Alternatively, use the known Fourier series for $\text{tri}(x)$: The even period 2π extension of f is

$$\tilde{f}_e(x) = 1 + \text{tri}(x) = 1 + \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos(nx)}{n^2}.$$

Problem 4. (15 points)
(a) (5) *Let $f(x) = x^2$ on the interval $[0, 1]$.*

What is the decay rate for the coefficients of Fourier cosine series for f?

Solution: The even period 2 extension of f has corners. So the coefficients decay like $1/n^2$.



(b) (5) *Let $f(t) = \cos(2t) + 2\cos(3t)$. Identify the smallest base period and corresponding fundamental angular frequency for f.*

Solution: Every frequency should be a multiple of the fundamental frequency. The greatest common divisor of 2 and 3 is 1, so the fundamental frequency is 1 and the base period is 2π.

Alternatively, $\cos(2t)$ has periods π, 2π, ... and $\cos(3t)$ has periods $2\pi/3$, $4\pi/3$, 2π, ... The smallest common period is 2π. So base period = 2π, fundamental frequency = 1.

(c) (5) *True or false: if the function $g(t)$ is periodic and even, then the periodic solution to $3x' + x = g(t)$ is also even. You must give a short justification for your answer.*

Solution: False. The phase lags can change a function from even to not even.