Name ____________________

# ES.1803 Quiz 6, Spring 2024

*3 problems, No books, notes or calculators.*

**There is a useful table on the last page of this quiz.**

**Problem 1.** (20 points)
**(a)** (15) Find the periodic solution to the DE $x'' + x' + 17x = 1 + \text{sq}(t)$, where $\text{sq}(t)$ is our usual odd, period $2\pi$, amplitude 1 square wave.

**(b)** (5) The period $2\pi$ impulse train has Fourier series $f(t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos(nt)$. Without solving the DE, say which term in the expansion of $f(t)$ causes the biggest response in the system $x'' + x' + 17x = f(t)$?

**Problem 2.** (30 points) For this problem we have a vibrating string of length $\pi$. The amplitude is given by $y(x,t)$, where $x$ is the position along the string and $t$ is time. The vibration is modeled by the wave equation

$$\text{PDE: } y_{tt}(x,t) = 9y_{xx}(x,t), \qquad 0 \le x \le \pi, \quad t \ge 0.$$

The ends of the string are set, so that they satisfy the boundary conditions

$$\text{BC: } y_x(0,t) = 0, \quad y_x(\pi,t) = 0.$$

Find the general solution to this PDE with these boundary conditions. You must show all the steps leading to the solution.

**Problem 3.** (25 points) Recall the PDE (heat equation) with BC:

PDE: $w_t(x,t) = w_{xx}(x,t), \quad t \ge 0, \quad 0 \le x \le \pi.$

BC: $w(0,t) = 0, \quad w(\pi,t) = 0.$

We have seen many times that the general solution to this is $w(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)e^{-n^2t}$.

**(a)** (10) Now consider the inhomogeneous PDE and BC:

IPDE: $u_t(x,t) = u_{xx}(x,t) - 2, \qquad 0 \le x \le \pi, \quad t \ge 0.$

IBC: $u(0,t) = 0, \quad u(\pi,t) = \pi^2.$

Give the general solution to IPDE and IBC:

Hint: Start by guessing a steady-state (time independent) particular solution.

**(b)** (10) Suppose that initially the rod has temperature $u(x,0) = x^2 + x$. Find the solution from Part (a) with these initial conditions.

**(c)** (5) Estimate the temperature of the rod at position $x = \pi/2$ at time $t = 1$, i.e., estimate $u(\pi/2, 1)$. You must give a finite numerical expression for the answer. Justify your answer, but you don't need to simplify the expression.

**Integrals** (for $n$ a positive integer)

1. $\displaystyle\int t\sin(\omega t)\,dt = \frac{-t\cos(\omega t)}{\omega} + \frac{\sin(\omega t)}{\omega^2}.$

2. $\displaystyle\int t\cos(\omega t)\,dt = \frac{t\sin(\omega t)}{\omega} + \frac{\cos(\omega t)}{\omega^2}.$

3. $\displaystyle\int t^2\sin(\omega t)\,dt = \frac{-t^2\cos(\omega t)}{\omega} + \frac{2t\sin(\omega t)}{\omega^2} + \frac{2\cos(\omega t)}{\omega^3}.$

4. $\displaystyle\int t^2\cos(\omega t)\,dt = \frac{t^2\sin(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\sin(\omega t)}{\omega^3}.$

1′. $\displaystyle\int_0^\pi t\sin(nt)\,dt = \frac{\pi(-1)^{n+1}}{n}.$

2′. $\displaystyle\int_0^\pi t\cos(nt)\,dt = \begin{cases} \dfrac{-2}{n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \neq 0 \text{ even} \end{cases}$

3′. $\displaystyle\int_0^\pi t^2\sin(nt)\,dt = \begin{cases} \dfrac{\pi^2}{n} - \dfrac{4}{n^3} & \text{for } n \text{ odd} \\ \dfrac{-\pi^2}{n} & \text{for } n \neq 0 \text{ even} \end{cases}$

4′. $\displaystyle\int_0^\pi t^2\cos(nt)\,dt = \frac{2\pi(-1)^n}{n^2}$

If $a \neq b$

5. $\displaystyle\int \cos(at)\cos(bt)\,dt = \frac{1}{2}\left[\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

6. $\displaystyle\int \sin(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\sin((a+b)t)}{a+b} + \frac{\sin((a-b)t)}{a-b}\right]$

7. $\displaystyle\int \cos(at)\sin(bt)\,dt = \frac{1}{2}\left[-\frac{\cos((a+b)t)}{a+b} + \frac{\cos((a-b)t)}{a-b}\right]$

8. $\displaystyle\int \cos(at)\cos(at)\,dt = \frac{1}{2}\left[\frac{\sin(2at)}{2a} + t\right]$

9. $\displaystyle\int \sin(at)\sin(at)\,dt = \frac{1}{2}\left[-\frac{\sin(2at)}{2a} + t\right]$

10. $\displaystyle\int \sin(at)\cos(at)\,dt = -\frac{\cos(2at)}{4a}$

**Some Fourier series:**

1. Period $2\pi$ square wave $\text{sq}(t)$: *You should know this for the quiz.*
2. Period 2 triangle wave $\text{tri2}(t)$:

   Over one period, $-1 \le t \le 1$, $\text{tri2}(t) = |t|$.

$$\begin{aligned}\text{tri2}(t) &= \frac{1}{2} - \frac{4}{\pi^2}\left(\cos(\pi t) + \frac{\cos(3\pi t)}{3^2} + \frac{\cos(5\pi t)}{5^2} + \cdots\right) \\ &= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{\cos(n\pi t)}{n^2}.\end{aligned}$$

*End of quiz*

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