ES.1803 Problem Section Problems for Quiz 7, Spring 2024 Solutions

This sheet contains problem section problems for **Topics 27-30**. Problem section problems for Topics 10-12 are in a separate file. Those for all other topics are posted with review for previous quizzes.

1 Systems of DEs

Topic 27 Phase portraits of linear 2×2 systems.

Problem 27.1. *Draw a phase protrait of* $\mathbf{x}' = A\mathbf{x}$*, where* $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$*. What type of critical point is at the origin? Is it dynamically stable?*

Solution: First we find the eigenvalues. The characteristic equation is

$$\lambda^2 - \text{tr}(A)\lambda + \det(A) = \lambda^2 + 4 = 0.$$

So the eigenvalues are $\pm 2i$. This means the critical point is a $\boxed{\text{center}}$.

The direction of rotation can be found by looking at the tangent vector at $(1, 0)$:

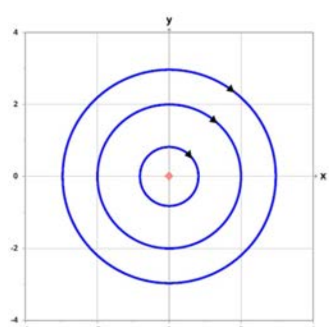
$$\mathbf{x}' = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

This tangent vector points down, which means that the ellipse is moving downwards at point $(1, 0)$ and so is moving clockwise.

Equivalently and more quickly: Because the $2, 1$ entry of A is negative, we know the trajectory turns in a $\boxed{\text{clockwise}}$ manner.

A center is on the boundary between dynamically asymptotically stable spiral sinks and dynamically unstable spiral sources. We call it an edge case. It is sometimes described as stable but not asymptotically stable.

For a center, when sketching a qualitative view of the phase portrait there is no need for eigenvectors. The trajectories are ellipses, which we have seen turn in a clockwise manner. For this system, the ellipses turn out to be perfect circles.

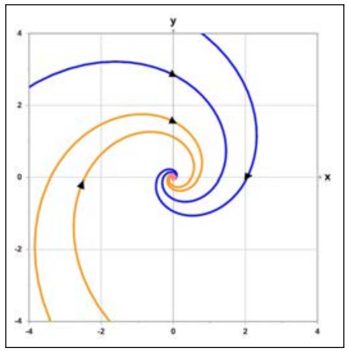


Problem 27.2. *Draw a phase protrait of* $\mathbf{x}' = A\mathbf{x}$*, where* $A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$*. What type of critical point is at the origin? Is it dynamically stable?*

Solution: The characteristic equation is $\lambda^2 + 2\lambda + 5 = 0$. So the eigenvalues are $-1 \pm 2i$.

Thus the critical point at the origin is a spiral sink. Since the 2, 1 entry of A is negative. The spiral turns in a clockwise manner. Spiral sinks are dynamically stable.

For spiral sinks, a qualitative phase portrait does not require computing the eigenvectors. By hand, we would just sketch clockwise spirals, spiraling in. (Of course, the graphing program we used here is more exact.)

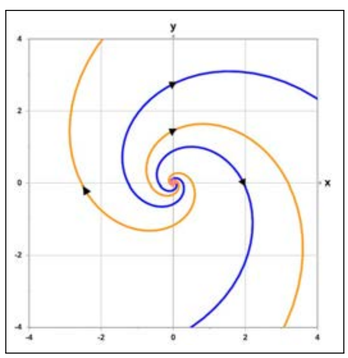


Problem 27.3. *Draw a phase protrait of* $\mathbf{x}' = A\mathbf{x}$, *where* $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. *What type of critical point is at the origin? Is it dynamically stable?*

Solution: The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$. So the eigenvalues are $1 \pm 2i$.

Thus the critical point at the origin is a spiral source. Since the 2, 1 entry of A is negative. The spiral turns in a clockwise manner. Spiral sources are dynamically unstable.

For spiral sources, a qualitative phase portrait does not require computing the eigenvectors. By hand, we just sketch clockwise spirals, spiraling out.



Problem 27.4. *Draw a phase protrait of* $\mathbf{x}' = A\mathbf{x}$, *where* $A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$. *What type of critical point is at the origin? Is it dynamically stable?*

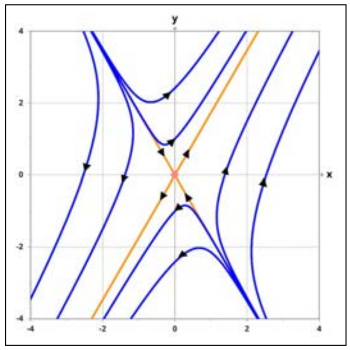
Solution: The characteristic equation is $\lambda^2 - 2\lambda - 2 = 0$. So the eigenvalues are $1 \pm \sqrt{3}$.

Since the eigenvalues have opposite signs, the critical point at the origin is a saddle. Saddles are dynamically unstable.

For saddles, a qualitative phase portrait requires computing the eigenvectors. We find that an eigenvector corresponding to $\lambda - 1 + \sqrt{3}$ is $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$, and one corresponding to $\lambda = 1 - \sqrt{3}$

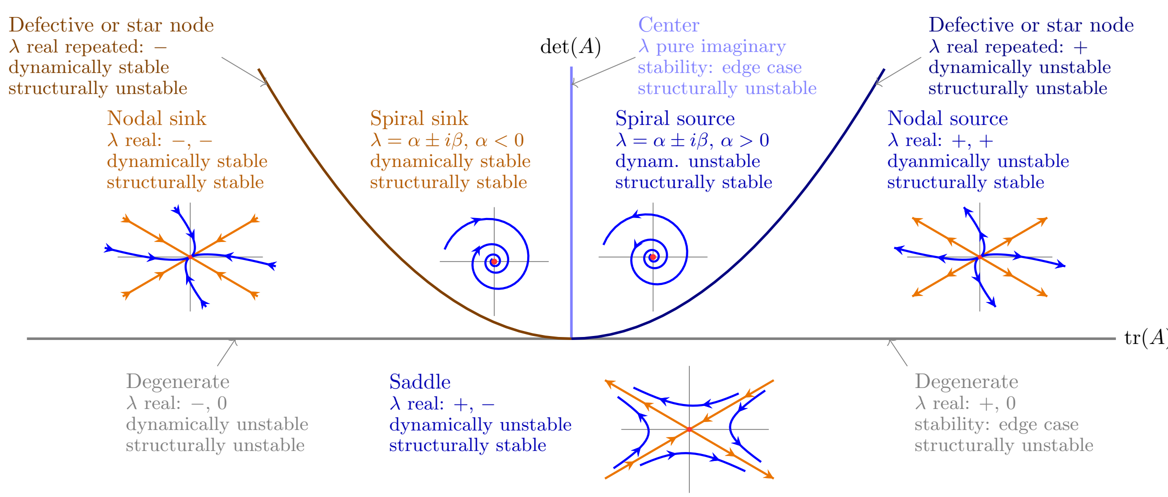
is $\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$

The modes give trajectories that are half lines. One mode has lines going out from the origin, and one has lines going in towards the origin. The mixed modal solutions are curves asymptotic to the modal trajectories at $t = \pm\infty$.



Problem 27.5. *Draw the trace-determinant diagram. Label all the parts with the type and dynamic stability of the critical point at the origin. Which types represent structurally stable systems?*

Solution: Here is the diagram:



The open regions in the diagram all represent structurally stable systems. That is, nodal sources, spiral sources, nodal sinks, spiral sinks and saddles are all structurally stable. The lines represent structurally unstable systems, i.e., defective and star nodes, centers, degenerate systems.

(b) *Give the equation for the parabola in the diagram. Explain where is comes from.*

Solution: The characteristic equation is $\lambda^2 - \text{tr}(A)\,\lambda + \det(A) = 0$. Therefore, the eigenvalues are

$$\lambda = \frac{\text{tr}(A) \pm \sqrt{\text{tr}(A)^2 - 4\det(A)}}{2}.$$

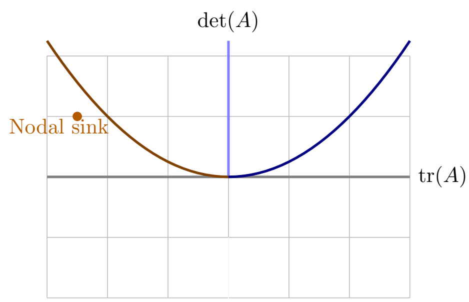
The parabola is the dividing line between real and imaginary root. That is it's where the discriminant (part under the square root) is 0. Its equation is

$$\operatorname{tr}(A)^2 - 4\det(A) = 0 \ \Leftrightarrow\ \det(A) = \frac{\operatorname{tr}(A)^2}{4}.$$

Problem 27.6. *Consider the linear system $\mathbf{x}' = A\mathbf{x}$.*

(a) *Suppose A has tr$(A) = -2.5$ and $\det(A) = 1$. Locate this system on the trace-determinant diagram. For this system, what is the type of the critical point at the origin?*

Solution: The diagram below shows the trace-determinant plane with the dividing lines included. The parabola has equation $\det(A) = \operatorname{tr}(A)^2/4$. The point $(-2.5, 1)$ is plotted. Since it is below the parabola in the third quadrant, it represents a nodal sink.



(b) *Compute the eigenvalues of this system and verify your answer in Part (a).*

Solution: The characteristic equation is

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = \lambda^2 + 2.5\lambda + 1 = 0.$$

Therefore, the eigenvalues are $\dfrac{-2.5 \pm \sqrt{6.25 - 4}}{2} = -0.5, -2$. Since these are real and negative, the critical point at the origin is a nodal sink. This matches the answer in Part (a).

Topic 28 Phase portraits of nonlinear 2×2 systems.

Problem 28.7. (a) *Sketch the phase portrait for $x' = -x + xy$, $y' = -2y + xy$.*

Solution: First we find the critical points by factoring the equations:

$x' = x(-1 + y) = 0 \Rightarrow x = 0$ or $y = 1$
$y' = y(-2 + x) = 0 \Rightarrow x = 2$ or $y = 0$

So the only critical points are $(0, 0)$ and $(2, 1)$.

Jacobian: $J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -1 + y & x \\ y & -2 + x \end{bmatrix}$

At (0,0): $J(0, 0) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

This is the coefficient matrix A of our linearized system as (0,0). The eigenvalues are -1, -2, so this is a linearized nodal sink. Since nodal sinks are structurally stable, we also have a nonlinear nodal sink.

(As an aside, it is worth noting that the eigenvectors lie along the axes and clearly there are trajectories along each axis, i.e., if $y = 0$ the trajecory is along the x-axis.)

At (2,1): $J(2,1) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$.

This is the coefficient matrix A of our linearized system as (2,1). The eigenvalues are $\pm\sqrt{2}$, so this is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

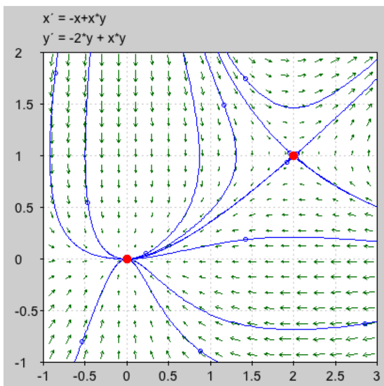
In order to sketch, we find the eigenvectors of the saddle:

The eigenvector equation is: $(A - \lambda I)\mathbf{v} = \mathbf{0}$,

$\lambda = \sqrt{2}$: $\quad A - \lambda I = \begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ \sqrt{2} \end{bmatrix}$.

$\lambda = -\sqrt{2}$: $\quad A - \lambda I = \begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ -\sqrt{2} \end{bmatrix}$.

Now we can sketch the linearized systems near each critical point and tie them together.



(b) *Consider x and y to be the sizes of two interacting populations. Tell a story about the populations.*

Solution: Alone each population has equation $x' = -x$ and $y' = -y$. So each would die off without the other. The interaction term xy is positive in both cases, so it seems these species cooperate to try to survive.

Unfortunately, it looks like there is a doomsday-extinction scenario. Depending on the initial conditions, Either the populations still die off to 0 (extinction) or else they explode to infinity (doomsday).

Problem 28.8. *Sketch the phase portrait for* $x' = x^2 - y, \quad y' = x(1 - y)$.

Draw one phase portrait for each possibility for the non-structurally stable critical point.

Solution: First we find the critical points.

Factoring the second equation: $y' = x(1 - y) = 0 \Rightarrow x = 0$ or $y = 1$.

Using these values in the second equation, $x' = x^2 - y = 0$ we find three critical points: $(0,0)$, $(1,1)$, $(-1,1)$.

Jacobian: $J(x,y) = \begin{bmatrix} 2x & -1 \\ 1-y & -x \end{bmatrix}$.

At $(0,0)$: $J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The eigenvalues are $\pm i$, so this is a linearized center. The 1 in the lower left entry of the matrix implies it turns counterclockwise.

Since centers are not structurally stable, we can't be sure the nonlinear system has a center at $(0,0)$. It could be a center, spiral source or spiral sink. We sketch all three possibilities below.

At $(1,1)$: $J(1,1) = \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}$. The eigenvalues are 2, -1, so this is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

In order to sketch, we find basic eigenvectors:

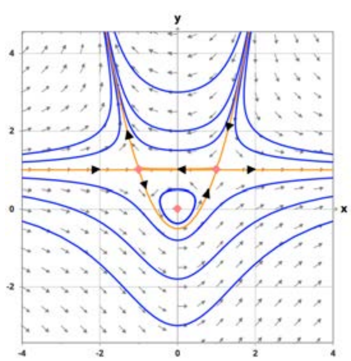
$\lambda = 2$: Take $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\quad \lambda = -1$: Take $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

At $(-1,1)$: $J(-1,1) = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix}$. The eigenvalues are -2, 1, so this is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

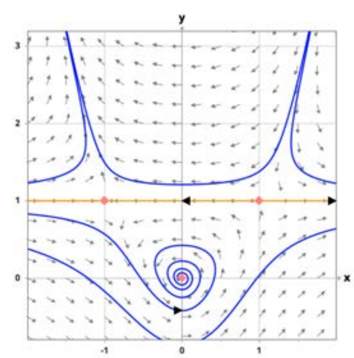
In order to sketch, we find basic eigenvectors:

$\lambda = -2$: Take $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\quad \lambda = 1$: Take $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

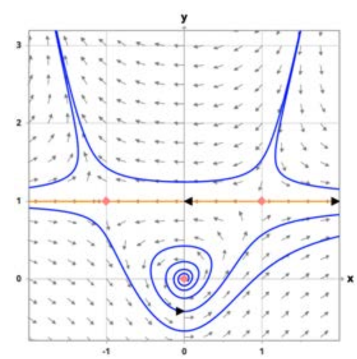
Here are sketches showing the three possible trajectories near the structurally unstable critical point.



Nonlinear center at (0,0)



Nonlinear spiral sink at (0,0)



Nonlinear spiral source at (0,0)

Problem 28.9. *Structural stability using the trace-determinant diagram: Will a non-structurally stable linearized critical point correctly predict the behavior of the nonlinear system at that point?*

Solution: Not necessarily. The linearized system is just an approximation of the nonlinear system. Non-structurally stable linearized systems might be qualitatively different from the nonlinear system they are approximating.

In the trace-determinant diagram the non-structurally stable systems are plotted on the boundary between different structurally stable systems.

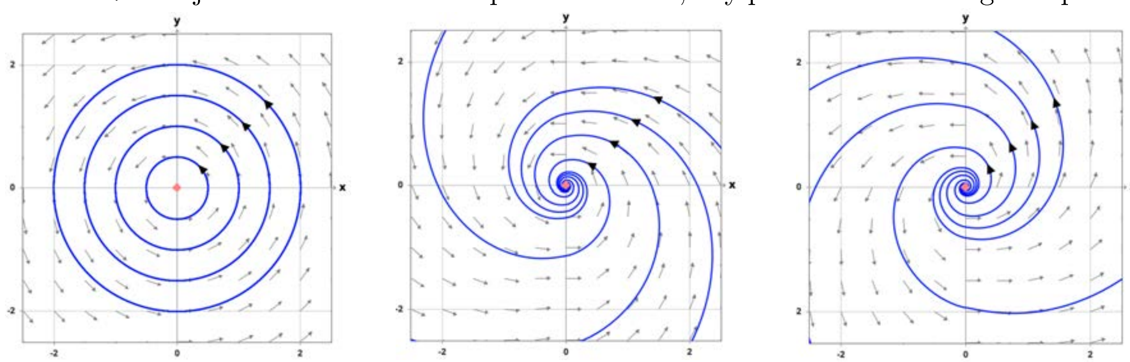
For example, a system with a linearized center at a critical point might be a nonlinear center or spiral. Here is a system which has a linearized center at the origin:

$$x' = -y + a \cdot x^3, \quad y' = x + a \cdot y \cdot x^2.$$

Clearly, for any value of a, the origin is a critical point. Also, for any a, $J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. So the origin is a linearized center (turning counterclockwise).

If $a = 0$, the system is linear and the critical point is a genuine center. If $a < 0$ the origin is a nonlinear spiral sink. If $a > 0$ it is a nonlinear spiral source.

For drawing, we actually used the system: $x' = -y + ax \cdot |x|^b$, $y' = x + ay|x|^b$. The parameter $b > 0$ is just there to make the spirals look nice, any positive value will gives spirals.



Center: $a = 0$ Sink: $a = -0.4$, $b = 0.2$ Source: $a = 0.04$, $b = 0.2$

Problem 28.10. *For the following system, draw the phase portrait by linearizing at the critical points.*

$$x' = 1 - y^2, \quad y' = x + 2y.$$

Solution: First we find the critical points.

$$x' = 1 - y^2 = 0 \Rightarrow y = \pm 1$$
$$y' = x + 2y = 0 \Rightarrow x = -2y.$$

So the only critical points are $(-2, 1)$ and $(2, -1)$.

Jacobian: $J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 0 & -2y \\ 1 & 2 \end{bmatrix}$

At (-2,1): $J(-2, 1) = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}$

This has eigenvalues are $1 \pm i$, so the critical point is is a linearized spiral source. The 1 in the lower left entry tells us it turns counterclockwise. Since spiral sources are structurally stable, we also have a nonlinear spiral source.

At (2,-1): $J(2, -1) = \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}$.

This has eigenvalues $1 \pm \sqrt{3}$, so this is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

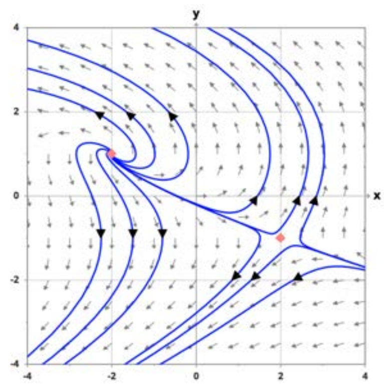
In order to sketch, we find the eigenvectors:

The eigenvector equation is: $(A - \lambda I)\mathbf{v} = \mathbf{0}$,

$\lambda = 1 + \sqrt{3}$: $\quad A - \lambda I = \begin{bmatrix} -1-\sqrt{3} & 2 \\ 1 & 1-\sqrt{3} \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ 1+\sqrt{3} \end{bmatrix}$.

$\lambda = 1 - \sqrt{3}$: $\quad A - \lambda I = \begin{bmatrix} -1+\sqrt{3} & 2 \\ 1 & 1+\sqrt{3} \end{bmatrix}$. Take $\mathbf{v} = \begin{bmatrix} 2 \\ 1-\sqrt{3} \end{bmatrix}$.

Now we can sketch the linearized systems near each critical point and tie them together.



Problem 28.11. *For the following system, draw the phase portrait by linearizing at the critical points.*

$$x' = x - y - x^2 + xy, \quad y' = -y - x^2.$$

Solution: First we find the critical points.

$$x' = x - y - x^2 + xy = 0$$
$$y' = -y - x^2 = 0.$$

The second equation implies $y = -x^2$. Putting this into the first equation gives

$$x + x^2 - x^2 - x^3 = x - x^3 = 0.$$

So, $x = 0,\ 1,\ -1$.

Thus the critical points are $(0,0)$, $(1,-1)$ and $(-1,-1)$.

Jacobian: $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 1-2x+y & -1+x \\ -2x & 1 \end{bmatrix}$

At (0,0): $J(0,0) = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$

This has eigenvalues are ± 1, so the critical point is is a linearized saddle. Since saddles are structurally stable, we also have a nonlinear saddle.

In order to sketch, we find the eigenvectors. This is straighforward, eigenvectors for $\lambda = 1, -1$ are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ respectively.

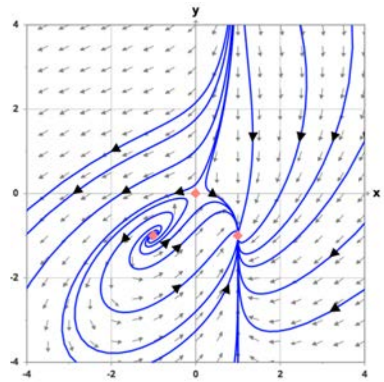
At (1,-1): $J(1,-1) = \begin{bmatrix} -2 & 0 \\ -2 & -1 \end{bmatrix}$.

This has eigenvalues -2, -1, so this is a linearized nodal sink. Since nodal sinks are structurally stable, we also have a nonlinear nodal sink.

At (-1,-1): $J(-1,-1) = \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}$.

This has eigenvalues $(1 \pm \sqrt{7}\,i)2$, so this is a linearized spiral source. The 2 in the lower left entry of the Jacobian tells us the spiral is counterclockwise. Since spiral sources are structurally stable, we also have a nonlinear spiral source.

Now we can sketch the linearized systems near each critical point and tie them together.



Problem 28.12. *Consider the system: $x' = x - 2y + 3$, $\quad y' = x - y + 2$.*

(a) *Find the one critical point and linearize at it. For the linearized system, what is the type of the critical point?*

Solution: The equations for the critical points are

$$\begin{aligned} x' &= x - 2y + 3 = 0 \\ y' &= x - y + 2 = 0. \end{aligned}$$

This is a linear system of equations. The only solution is $(x, y) = (-1, 1)$.

Jacobian: $J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

So, $J(-1,1) = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$. Thus the linearized system at the critical point is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

This has characteristic equation $\lambda^2 + 1 = 0$. So the eigenvalues are $\pm i$. This shows the linearized system is a center.

(b) *In Part (a) you should have found that the linearized system is a center. Since this is not structurally stable, it is not necessarily true that the nonlinear system has a center at the critical point. Nonetheless, in this case, it does turn out to be a nonlinear center. Prove this.*

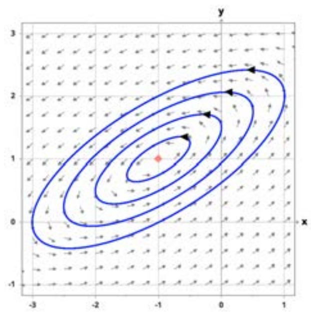
Solution: This is an inhomogeneous linear system with constant input. So one way to make a phase portrait is to solve the equation and plot trajectories.

Since the input is constant, we guess a constant solution $\mathbf{x} = \mathbf{K}$. We find $\mathbf{x_p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. (Not surprisingly this is the same as the critical point!)

The associated homogeneous system is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. For the linear, homogeneous system, the coefficient matrix has eigenvalues $\pm i$. Thus the critical point at the origin is a center.

The general solution is $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$.

Since $\mathbf{x_p}$ is a constant, the general inhomogeneous solution is just the homogeneous solution translated by $(-1, 1)$. This shows that the critical point at $(-1, 1)$ is, indeed, a center.



Topic 29: Structural stability

This will be covered in topic 28: nonlinear phase portraits.

Topic 30 Population models

Problem 30.13. *Let $x(t)$ be the population of sharks off the coast of Massachusetts and $y(t)$ the population of fish. Assume that the populations satisfy the Volterra predator-prey equations*

$$x' = ax - pxy; \qquad y' = -by + qxy, \qquad \text{where } a,\ b,\ p,\ q, \text{ are positive.}$$

Assume time is in years and a and b have units $1/\text{years}$.

Suppose that, in a few years, warming waters start killing 10% of both the fish and the sharks each year. Show that the shark population will actually increase.

Solution: Original equations:

$$\begin{aligned} \text{sharks: } x' &= ax - pxy \\ \text{fish: } y' &= -by + qxy. \end{aligned}$$

The original equilibrium is (sharks, fish) $= (\frac{b}{q}, \frac{a}{p})$.

With warming:

$$x' = (a - 0.1)x - pxy$$
$$y' = -(b + 0.1)y + qxy$$

The new equilibrium is (sharks, fish) $= \left(\frac{b+0.1}{q}, \frac{a-0.1}{p}\right)$. So the equilibrium level of sharks increases. (And that of fish decreases.)

Problem 30.14. *Consider the system of equations*

$$x'(t) = 39x - 3x^2 - 3xy; \qquad y'(t) = 28y - y^2 - 4xy.$$

The four critical points of this system are (0,0), (13,0), (0,28), (5,8).

(a) *Show that the linearized system at (0,0) has eigenvalues 39 and 28. What type of critical point is (0,0)?*

Solution: The Jacobian of the system is $J(x, y) = \begin{bmatrix} 39 - 6x - 3y & -3x \\ -4y & 28 - 2y - 4x \end{bmatrix}$.

(a) $J(0, 0) = \begin{bmatrix} 39 & 0 \\ 0 & 28 \end{bmatrix}$. This is a diagonal matrix, so the eigenvalues are the diagonal entries: $\lambda = 39$, 28. Positive real eigenvalues imply the linearized critical point is a nodal source. This is structurally stable, so the nonlinear critical point is also a nodal source.

(b) *Linearize the system at (13,0); find the eigenvalues; give the type of the critical point.*

Solution: $J(13, 0) = \begin{bmatrix} -39 & -39 \\ 0 & -24 \end{bmatrix}$. This is triangular, so the eigenvalues are just the diagonal entries: $\lambda = -39$, -24. Negative eigenvalues imply the linearized critical point is a nodal sink. This is structurally stable, so the nonlinear critical point is also a nodal sink.

(c) *Repeat Part (b) for the critical point (0,28).*

Solution: $J(0, 28) = \begin{bmatrix} -45 & 0 \\ -112 & -28 \end{bmatrix}$. This is triangular, so the eigenvalues are just the diagonal entries: $\lambda = -45$, -28. Negative eigenvalues imply the linearized critical point is a nodal sink. This is structurally stable, so the nonlinear critical point is also a nodal sink.

(d) *Repeat Part (b) for the critical point (5,8).*

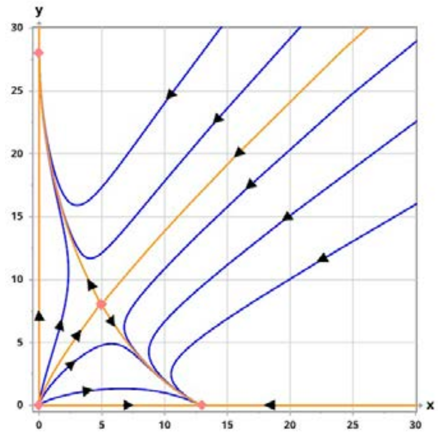
Solution: $J(5, 8) = \begin{bmatrix} -15 & -15 \\ -32 & -8 \end{bmatrix}$. The characteristic equation is $\lambda^2 + 23\lambda - 360 = 0$. This has eigenvalues $\dfrac{-23 \pm \sqrt{23^2 + 4 \cdot 360}}{2}$. That is, it has one positive and one negative eigenvalue. Therefore, the linearized critical point is a saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

Note: We could also have identified this as a saddle because its determinant is negative.

(e) *Sketch a phase portrait of the system. If this models two species, what is the relationship between the species? What happens in the long-run?*

Solution: Here is the phase portrait. Note the separatix (in orange). It is made up of the trajectories that go asymptotically to the saddle point (5,8).

The relationship between the species is one of competition –you see that because both x' and y' have a $-xy$ term. In the long run one species dies out and the other stabilizes at the carrying capacity of the environment.



Problem 30.15. *The system for this equation is*

$$x' = 4x - x^2 - xy$$
$$y' = -y + xy$$

(a) *This models two populations with a predator-prey relationship. Which variable is the predator population?*

Solution: In the presence of y, the growth rate of x decreases. In the presence of x, the growth rate of y increases. Thus x is the prey population and y the predator population.

(b) *What would happen to the predator population in the absence of prey? What about the prey population in the absence of predators?*

Solution: Without prey, i.e., when $x = 0$, the DE for y is $y' = -y$. This is exponential decay. So eventually the predator population would go to 0.

Without predators, the equation for the prey becomes $x' = 4x - x^2$. This is the logistic equation with dynamically stable critical point $x = 4$ and dynamically unstable critical point $x = 0$. The prey population would eventially stabilize at 4.

(c) *There are three critical points. Find and classify them*

Solution: We can factor each of the equations to find the critical points:

$$x' = x(4 - x - y) = 0 \Rightarrow x = 0 \text{ or } 4 - x - y = 0$$
$$y' = y(-1 + x) \Rightarrow y = 0 \text{ or } x = 1.$$

The critical points are $(0, 0)$, $(4, 0)$, $(1, 3)$.

The Jacobian is $J(x, y) = \begin{bmatrix} 4 - 2x - y & -x \\ y & -1 + x \end{bmatrix}$.

Considering each of the critical points in turn:

$J(0, 0) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \lambda = 4, -1.$

One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(4,0) = \begin{bmatrix} -4 & -4 \\ 0 & 3 \end{bmatrix} \Rightarrow \lambda = -4, 3.$

One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(1,3) = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}.$

Characteristic equation: $\lambda^2 + \lambda + 3 = 0 \quad \Rightarrow \lambda = -1 \pm \sqrt{11}\, i.$

Complex eigenvalues with negative real part imply this is a linearized spiral sink. This is structurally stable, so the nonlinear critical point is also a spiral sink.

(d) *Sketch a phase portrait of this system. What is the relationship between the species? What happens in the long-run?*

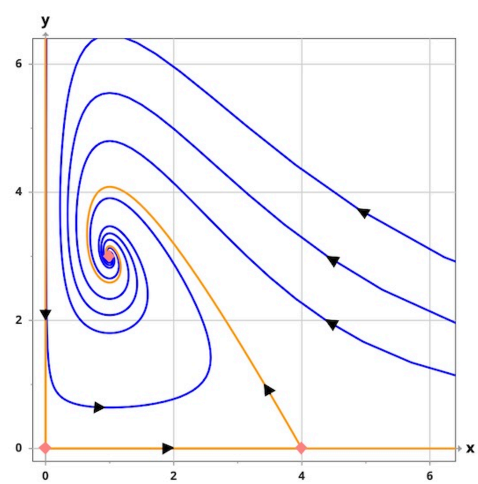
Solution: For the saddles, we need to find the eigenvectors. For the spiral, we need its direction.

$J(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ has independent eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

$J(4,0) = \begin{bmatrix} -4 & -4 \\ 0 & 3 \end{bmatrix}$ has independent eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \end{bmatrix}.$

The spiral at $(1,3)$ is counterclockwise because of the 3 in the lower left entry of $J(1,3)$.

Here is the phase portrait. Since we're talking about populations, the portrait only shows the first quadrant. All trajectories spiral into the critical point at (2,3). (Actually, there are a handful of trajectories along the axes that go asymptotically to the saddle points.)

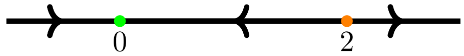


Problem 30.16. *The equations for this system are*

$$x' = x^2 - 2x - xy$$
$$y' = y^2 - 4y + xy$$

(a) *If this models two populations, what would happen to each of the populations in the absence of the other?*

Solution: If $y(t) = 0$, then $x' = x^2 - 2x$. This has critical points $x = 0, 2$ and phase line



So, without any predator ($y(t) = 0$), the prey population x will either crash to 0 or boom to infinity –at least according to this model.

The answer is the same for $y(t)$ if $x(t) = 0$.

(b) *There are four critical points. Find and classify them*

Solution: Again, we can factor to find the critical points.

$$x' = x(x - 2 - y) = 0 \;\Rightarrow\; x = 0 \text{ or } x - 2 - y = 0$$
$$y' = y(y - 4 + x) = 0 \;\Rightarrow\; y = 0 \text{ or } y - 4 - x = 0.$$

First let $x = 0$, then $y = 0$ or $y = 4$: two critical points (0,0), (0,4).

Next let $y = 0$, then $x = 0$ or $x = 2$: one more critical point (2,0).

Finally, solve $x - 2 - y = 0$, $y - 4 - x = 0$: one more critical (3,1).

The Jacobian is $J(x, y) = \begin{bmatrix} 2x - 2 - y & -x \\ y & 2y - 4 + x \end{bmatrix}$. Looking at each critical point in turn we get

$J(0, 0) = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \;\Rightarrow\; \lambda = -2, -4$. Negative eigenvalues imply this is a linearized nodal sink. This is structurally stable, so the nonlinear critical point is also a nodal sink.

$J(0, 4) = \begin{bmatrix} -6 & 0 \\ 4 & 4 \end{bmatrix} \;\Rightarrow\; \lambda = -6, 4$. One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(2, 0) = \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix} \;\Rightarrow\; \lambda = 2, -2$. One positive and one negative eigenvalue imply this is a linearized saddle. This is structurally stable, so the nonlinear critical point is also a saddle.

$J(3, 1) = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$.

Characteristic equation: $\lambda^2 - 4\lambda + 6 = 0 \;\Rightarrow\; \lambda = 2 \pm \sqrt{2}\, i$ Complex eigenvalues with positive real part imply this is a linearized spiral source. This is structurally stable, so the nonlinear critical point is also a spiral source.

(c) *Sketch a phase portrait of the system. What is the relationship between the species? What happens in the long-run?*

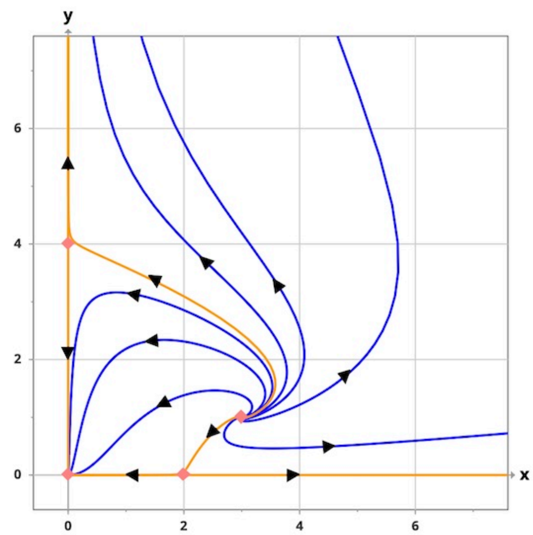
Solution: For the saddles, we need to find the eigenvectors. For the spiral, we need its direction.

$J(0, 4) = \begin{bmatrix} -6 & 0 \\ 4 & 4 \end{bmatrix}$ has independent eigenvectors $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$J(2,0) = \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix}$ has independent eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The spiral at $(1,3)$ is counterclockwise because of the 1 in the lower left entry of $J(1,3)$.

Here is the phase portrait. The trajectories either go asymptotically to $(0,0)$ or to ∞. This looks like a predator-prey relationship. What seems more important, is that each population by itself is modeled by a doomsday-extinction equation. That is, either the population goes to ∞ or to 0. It's hard to tell exactly, but it seems that when the predator (y) goes to infinity, the prey (x) goes extinct.



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2 First-order nonlinear

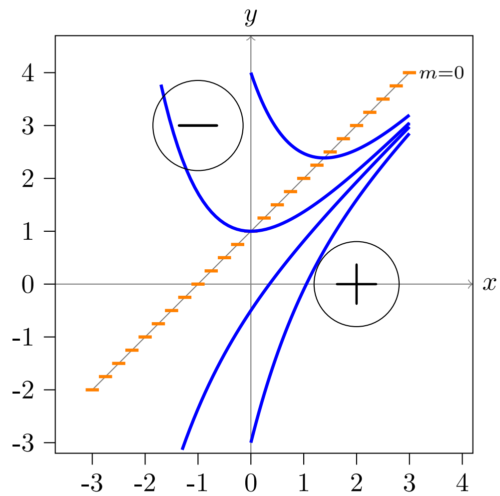
Topic 10: Direction fields, integral curves, existence of solutions

Problem 10.17. *Consider $y' = x - y + 1$.*

(a) *Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as + and negative slope as −. Use this to give a very rough sketch of some solution curves.*

Solution: The nullcline is the isocline with $m = 0$. In our case, this is $0 = x - y + 1$, which we also write as $y = x + 1$. Above this line, we have $y > x + 1$ or $0 > x - y + 1$, which means the slope is negative. Below this line, we have $y < x + 1$ and $0 < x - y + 1$ so slope is positive.

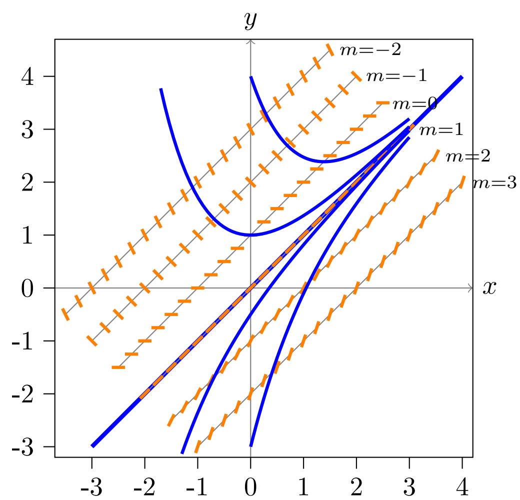
We draw horizontal lines along the nullcline to indicate the slope. The slope field is negative above the line, so integral curves in this region go down towards the nullcline, level off to slope 0 at the nullcline and then turn upwards. The slope field is positive below the nullcline, so integral curves in this region all slope upwards.



(b) *Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.*

(Note the isocline $y = x$ happens to be a solution—don't expect this to happen usually.)

Solution: See graph:



(c) *Can you make a squeezing argument that shows that all solutions go asymptotically to the line $y = x$.*

Solution: This is a little tricky since we will use an indirect argument. We'll consider integral curves below the integral curve $y = x$. The argument for those above $y = x$ is similar.

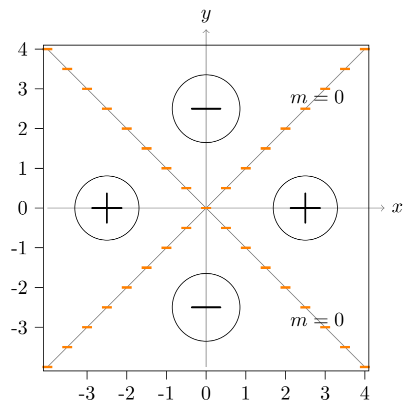
The isoclines are all parallel to the integral curve $y = x$. That is, as lines they have slope 1. The isoclines below the line $y = x$ all have slope field elements of slope greater than 1. The slope of an integral curve below $y = x$ must go asymptotically to 1. (If it stayed greater than $1 + b$, for some positive b, then it would have to keep growing faster than the line $y = x$ and, therefore, cross $y = x$.) If the slope of an integral curve goes asymptotically to 1, the curve must approach the isocline with $m = 1$, i.e. $y = x$.

Problem 10.18. *Consider $y' = x^2 - y^2$*

(a) *Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as + and negative slope as −. Use this to give a very rough sketch of some solution curves.*

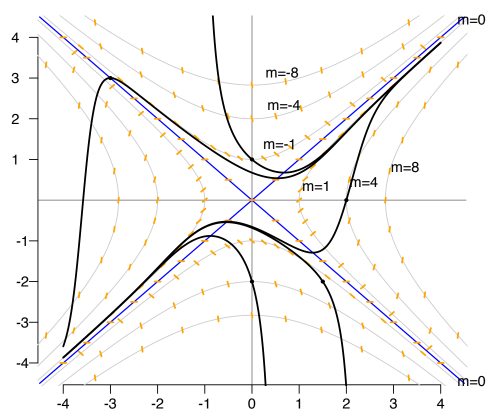
Note: the nullcline consists of two lines.

Solution: The nullcline consists of the lines $y = \pm x$. Below is a sketch of the nullcline with the regions marked + or −. Look at the figure with Part (b) for some integral curves.



(b) *Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.*

Solution: Isoclines are hyperbolas with asymptotes $y = \pm x$.



(c) *Add some integral curves to the plot in Part (b). Include the one with $y(2) = 0$.*

Solution: See plot in Part (b).

(d) *Use squeezing to estimate $y(100)$ for the solution with IC $y(2) = 0$.*

Solution: We can see from the plot in Part (b) that this solution seems to go asymptotically to the nullcline $y = x$.

The argument to see this is a little subtle. We'll give the argument as a sequence of observations. On an exam, you could just state this as an empirical observation about the isoclines sketch.

1. Clearly the nullcline $y = x$ is an upper fence for this integral curve, so the curve stays

below this line.

2. To be specific, let's take $m = 2$. The isocline for $m = 2$ goes asymptotically to the line $y = x$. That is, its slope as a curve (not the isocline slope) is close to 1 for large x. Thus, when x is large, the isocline $m = 2$ is a lower fence, i.e., its slope element goes from below to above the isocline.

3. Let x be large enough that the isocline for $m = 2$ is a lower fence. If the integral curve $y(x)$ is below the isocline then its slope is bigger than 2. This means it is growing faster than the isocline and must eventually cross it. At this point it is above the fence and in the funnel between $y = x$ and the isocline for $m = 2$.

4. This funnel goes asymptotically to $y = x$, so we can estimate $y(100) \approx 100$.

Problem 10.19. *Consider $y' = y(1-y)$ (Note that there is no x; what does this mean for the shape of your nullclines? Your isoclines?)*

(a) *Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as + and negative slope as −. Use this to give a very rough sketch of some solution curves.*

Solution: Note. This is secretly introducing autonomous equations.

The sketch is shown with the solution to Part (b).

The nullcline is $0 = y(1-y)$. For this equation to work, we need either $y = 0$ or $y = 1$. Therefore, these two lines are our nullclines. Since the tangent elements lie along the lines, the nullclines turn out to be solutions.

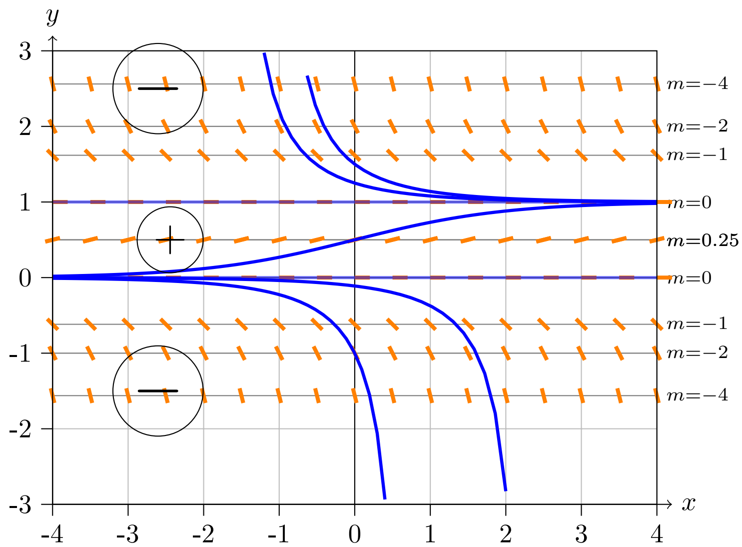
We can see that for $y > 1$, $y' = y(1-y) < 0$, i.e. the slope field is negative. Likewise, for $0 < y < 1$, $y > 0$, so the slope field is positive. Finally, for $y < 0$, $y' < 0$, so the slope field is negative.

Since the nullclines are solutions, no other solutions cannot cross them. This means each solution curve is restricted to one section of the graph. So we have integral curves that come down from $y = \infty$ and go asymptotically to the top nullcline. Likewise, we have integral curves coming up from $y = -\infty$ and going asymptotically to the bottom nullcline. Finally, in the middle section, we have integral curves that come asymptotically from the bottom nullcline and go asymptotically up to the top one. These have a flat S-shape with positive slope. All solutions repeat identically when translated in the horizontal direction.

See the graph in Part (b). Notice that we could come close to drawing it knowing just the nullclines.

(b) *Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.*

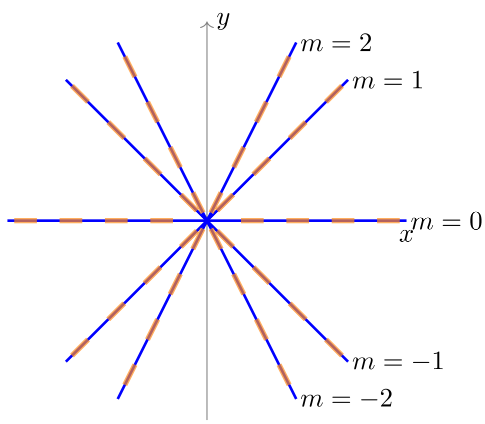
We can see that all isoclines are horizontal lines because the equation for y' does not depend on x and so is constant when y is fixed and x changes.



Problem 10.20. *Consider $y' = y/x$ Note: the line $x = 0$ ($m = \infty$) also separates regions of positive and negative slope.*

(a) *Sketch the isoclines for $m = 0, \pm 1, \pm 2$. Use it this to give a sketch of some solutions.*

Solution: The isoclines are $y/x = m$ or $y = mx$. These are lines that happen to have the same slope as the slope field elements along them. This shows that each of these isoclines is actually a solution.



(b) *This is a rare case where we can solve the DE. Solve the DE and use your solution to draw some integral curves.*

Solution: Separating variables: $\frac{dy}{y} = \frac{dx}{x}$.

Integrating: $\ln|y| = \ln|x| + C$.

Solving for y: $y = Cx$. All solutions are lines through the origin.

Picture is the same as in Part (a).

Note: Because there are no solutions that go through points on the y-axis (other than $(0, 0)$), existence of solutions through every point fails. Also, because there are many solutions that go through the origin, uniqueness fails. This is not surpising since $f(x, y) = y/x$ is not continuous when $x = 0$.

Problem 10.21. *For $y' = -y/(x^2 + y^2)$, sketch the direction field in the upper half-plane. For the solution with initial condition $y(0) = 1$ explain why you know it is decreasing for $x > 0$. Explain why it is always positive for $x > 0$.*

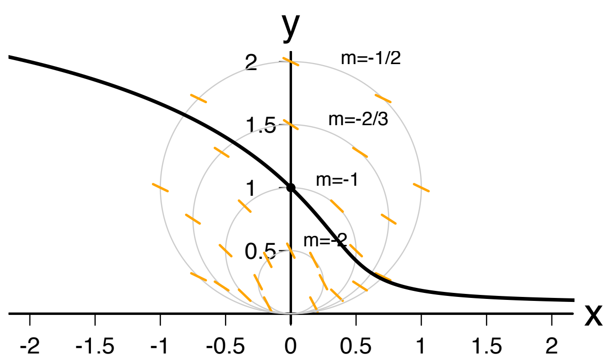
Solution: The isoclines are $-y/(x^2 + y^2) = m$. A little algebra converts this to the form

$$x^2 + \left(y + \frac{1}{2m}\right)^2 = \frac{1}{4m^2}.$$

This is a circle of radius $1/(2m)$ centered at the point $(0, -1/(2m))$ on the y-axis. Note that these all go through the origin. This is okay since $-y/(x^2 + y^2)$ is not defined at the origin, so all bets are off there.

The isoclines with negative slope $(m < 0)$ are in the upper-half plane. We know that because $y' = -y/(x^2 + y^2)$ is negative when $y > 0$.

The solution will always be positive because we know that $y(x) = 0$ is a solution (just plug it into the DE). Except at the origin, the existence and uniqueness theorem guarantees that integral curves don't cross. This means an integral curve that starts positive can't cross $y = 0$, so it must stay positive.

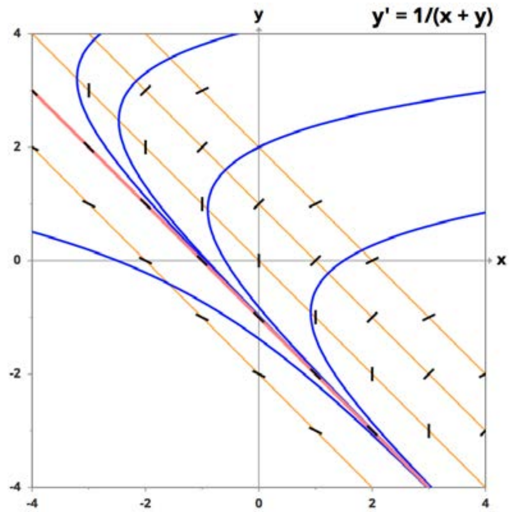


Problem 10.22. *Consider the DE $y' = \dfrac{1}{x + y}$*

Draw a direction field by using about five isoclines; the picture should be square, using the intervals between -4 and 4 on both axes.

Sketch in the integral curves that pass respectively through (0,0), (-1,1), (0,-2). Will these curves cross the line $y = -x - 1$? Explain by using the existence and uniqueness theorem

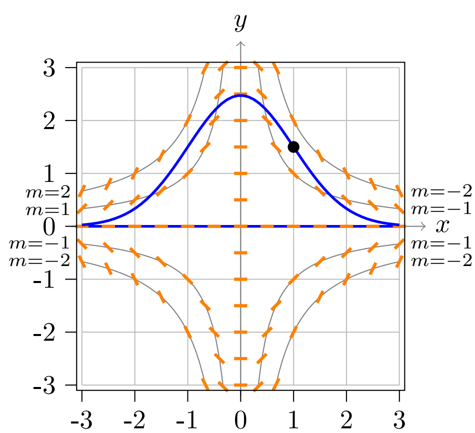
Solution: The isocline for slope m is $\dfrac{1}{x + y} = m$. For $m \neq 0$ this is equivalent to $x + y = 1/m$. These are lines of slope -1. Several are shown in the figure below. The isocline with $m = -1$ is also an integral curve (its slope field elements are all along the line). Since $f(x, y) = 1/(x + y)$ is continuous along the line $x + y = -1$, the existence and uniqueness theorem guarantees that other integral curves can't cross it.



Problem 10.23. *Consider the DE $y' = -xy$.*

(a) *Draw a direction field using isoclines for $m = 0,\ 1,\ 2,\ -1,\ -2$.*

Solution: The nullcline consists of both axes. The isoclines are hyperbolas with two branches, asymptotic to the axes.



(b) *Let $y(x)$ be the solution with initial condition $y(1) = 1.5$. Use fences and funnels to estimate $y(100)$.*

Solution: The x-axis is an integral curve for the solution $y(x) = 0$ This acts as a fence. The isocline for $m = -2$ is an upper fence when $x > 1$. (This is because the slope elements go from above to below the isocline.) Together these two fences form a funnel that goes asymptotically to $y = 0$.

Since the initial point $(1, 1.5)$ is inside the funnel we can estimate $y(100) \approx 0$. The exact value will be slightly bigger.

Topic 11: Numerical methods

Problem 11.24. *For $y' = y^2 - x^2$:*

(a) *Use Euler's method with $h = 0.5$ to estimate $y(3)$ for the solution with initial condition $y(2) = 0$.*

Solution: As in the Topic 11 notes, set up a table with columns: n, x_n, y_n, m, mh.

n	x_n	y_n	m	mh
0	2	0	-4	-2
1	2.5	-2	-2.25	-1.125
2	3	-3.125		

(b) *Is the estimate in Part (a) too high or too low?*

Solution: We can take the derivative of our equation to get the equation for the second derivative $y'' = 2yy' - 2x$. If we look at the point $(x, y) = (2, 0)$, then we can use our original equation to get $y' = -4$, and the second derivative equation to get $y'' = -4 < 0$. A negative second derivative implies the integral curve is concave down, which implies that our estimate is an overestimate, since drawing tangent lines to the curve produces values above the curve.

Problem 11.25. *For $\frac{dy}{dx} = F(x, y) = y^2 - x^2$.*

(a) *Use Euler's method to estimate the value at $x = 1.5$ of the solution for which $y(0) = -1$. Use step size $h = 0.5$. As in the notes, make a table with columns n, x_n, y_n, m, mh.*

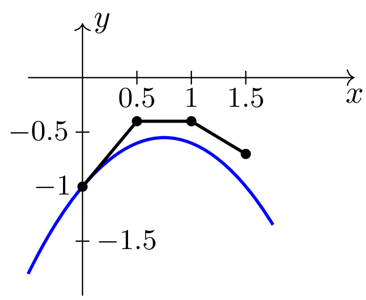
We are estimating y(1.5) using Euler's method with step size 0.5. This takes $\frac{1.5-0}{0.5} = 3$ steps, as outlined in the following table.

n	x_n	y_n	m_n	$m_n h$
0	0	-1	1	0.5
1	0.5	-0.5	0	0
2	1.0	-0.5	-0.75	-0.375
3	1.5	-0.875		

Thus Euler's method gives the estimate

$$y(1.5) \approx y_3 = -0.875.$$

The corresponding Euler polygon for this estimation is



Euler polygon and actual integral curve.

(b) *Is the estimate found in Part (a) likely to be too large or too small?*

It is likely to be too large. One way to see this is to use the second derivative test to check the concavity of solutions around the inital point. First, find $\frac{d^2y}{dx^2} = y''$ by taking the derivative of the differential equation:

$$y'' = \frac{d}{dx}(F(x,y)) = \frac{d}{dx}(y^2 - x^2) = 2yy' - 2x,$$

Second, evaluate the second derivative at the initial point $(0, -1)$ to get

$$y''|_{(0,-1)} = 2(-1)(1) - 2(0) = -2 < 0.$$

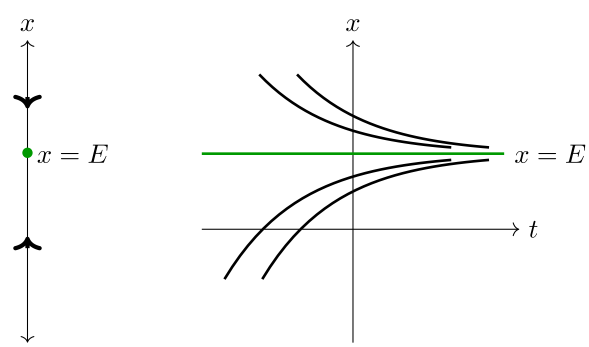
This means the solution that goes through the initial point is concave down. The tangent to a concave down function lies above the function in a small neighborhood, so the Euler estimate for one step is likely to overshoot. Running the same check for the next two endpoints, shows that the second derivative is negative at each endpoint of the Euler polygon. So each of the three steps is likely to overshoot. This suggests the estimate found is likely to be greater than the value of the true solution when $x = 1.5$.

Topic 12: Automonous DEs and bifurcation diagrams

Problem 12.26. *For the following DE, find the critical points, draw the phase line, sketch some integral curves, 'explain' the model.*

Temperature: $x' = -k(x - E)$ *(E constant ambient temperature).*

Solution: Critical points are when $f(x) = 0$, which in this case is just $x = E$. To draw a phase line, we see that x' is negative for $x > E$ and x' is positive for $x > E$. Looking at the phaseline, we see that $x = E$ is a stable critical point.



If we draw some integral curves, we get curves that head toward the line $x = E$. The intuitive explanation behind this model is that the temperature x heads towards the ambient temperature.

Problem 12.27. *Suppose the following DE models a population* $x' = -ax + 1$, *which is a constant birth-and-death rate situation modified to include a constant rate of replenishment.*

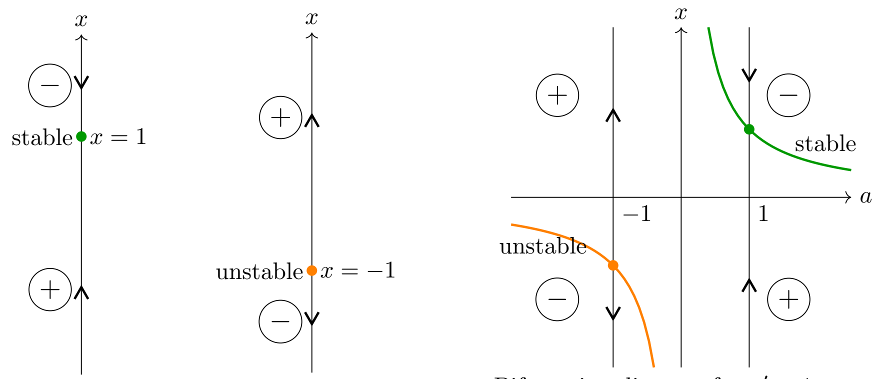
(i) Sketch the bifurcation diagram and list any bifurcation points (these are special values of a*).*

(ii) The bifurcation points divide the a-axis into intervals. Illustrate one typical case for each interval by giving the phase line diagram. For each of these phase lines, give (rough) sketches of solutions in the tx-plane.

(iii) For what values of a is the population sustainable. What happens for other values of a.

Note the applet 'phase lines' can show this system.

Solution: We answer (i) and (ii) together. The critical points are $x' = -ax + 1 = 0$. So, $x = 1/a$. We graph this in the ax-plane –it's a hyperbola with two branches. Here is the finished bifurcation diagram with two phase lines. These are explained below.



Bifurcation diagram for $x' = 1 - ax$

After plotting the critical points, we see that the graph divides the ax-plane into 3 regions. In order to determine the sign of x' in each region, we found phase lines for $a = 1$ and $a = -1$. These are shown at the left. Determining the direction of the arrows was straightforward and we leave it to the reader to supply the details.

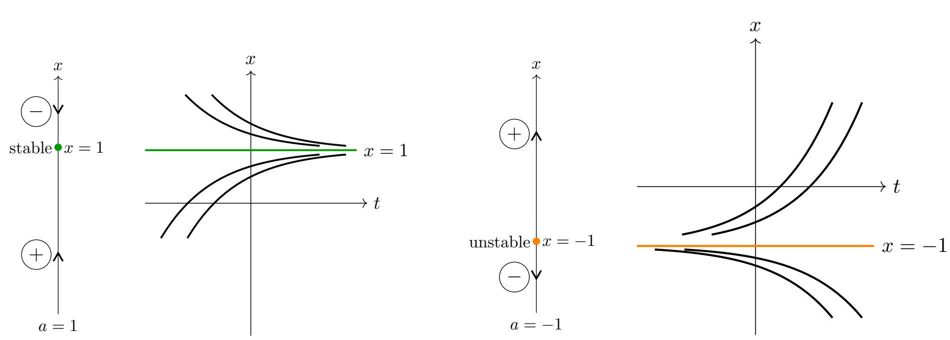
We place the phase lines on the bifurcation diagram at $a = 1$ and $a = -1$. (This answers (ii).) The arrows on the phase lines then tell us the sign of x' in all 3 regions.

Once we know the sign on x', it's a simple matter to decide the stability of each part of the diagram. The stable branch is drawn in green and labeled 'stable'. Likewise, the unstable branch is drawn in orange and labeled 'unstable'.

There is one bifurcation point at $a = 0$. This is a bifurcation point because the bifurcation diagram is qualitatively different on either side of $a = 0$.

(iii) When $a > 0$ there is a positive stable equilibrium, so the population is sustainable. When $a \leq 0$ the population is not sustainable. In fact, it blows up to infinity.

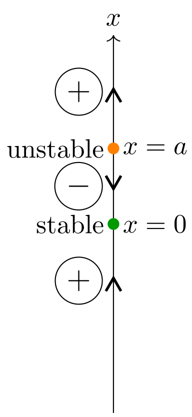
Finally, we do our duty and sketch some solution curves based on the phase lines.



Problem 12.28. *Consider the system $x' = x(x-a) + \frac{1}{4}$, which is the 'doomsday-vs-extinction' equation with the addition of a constant rate of replenishment.*

(a) *First consider the equation $x' = x(x-a)$ with $a > 0$. Why is this called the doomsday-vs-extinction population model?*

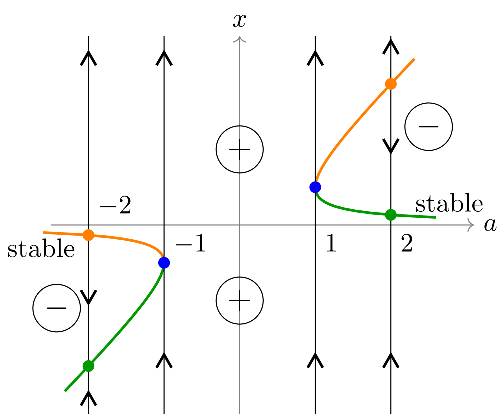
Solution: We draw the phase line. The critical points are $x = 0$ and $x = a$. It is easy to determine the sign of x', these are indicated by the arrows on the phase line.



We see that there is no positive stable equilibrium. If x starts greater than a, then it will increase to infinity –doomsday. If x starts less than a, then it will decrease to 0 –extinction.

(b) *Sketch the bifurcation diagram for $x' = x(x-a) + 1/4$.*

Solution: Here is the bifurcation diagram. It requires a bit of calculus to graph properly and figure the values of various points. This is explained below.



Bifurcation diagram for $x' = x(x-a) + 1/4$

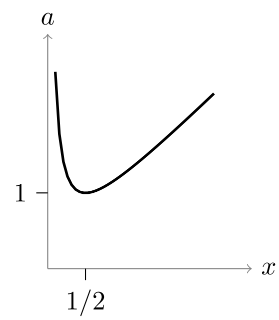
Here is the explanation for the bifurcation diagram.

Computing the crititcal points is simple algebra

$$x' = x(x-a) + \frac{1}{4} = 0 \quad \Rightarrow \quad a = x + \frac{1}{4x}.$$

First notice that this gives a as a function of x. So we'll first plot it with the axes reversed and just for positive x. When x is small a is large. When x is large $a \approx x$. This leads to the graph shown below. We can use calculus to find that the minimum is at $x = 1/2$, $a = 1$.

(That is, we solve $\dfrac{da}{dx} = 0$.)



To get the bifurcation diagram we just interchange the axes. Free of charge, we see that $a = 1$ is the smallest positive value of a on the diagram.

The part of the diagram with $x < 0$ is found similarly.

We then plotted a number of phase lines to identify the regions where x' is positive and negative. As usual, these are marked with $+$ or $-$. Using these, we can identify the stable and unstable parts of the bifurcation diagram.

(c) *Identify the bifurcation points. For what values of a is the population sustainable? Which positive values of a guarantee against extinction? Which positive values of a guarantee against doomsday?*

Solution: The bifurcation points are at $a = \pm 1$.

There are positive stable equilibrium for $a > 1$, so this is the range of a where the population is sustainable.

For $a > 1$, the population either stabilizes at the positive stable equilibrium or blows up to infinity. So, for $a > 1$, x won't go extinct.

For $a = 1$, x will not linger at the semistable equilibrium, instead it is likely to blow up.

For $0 < a < 1$, the population always blows up to infinity. So, for these a, x won't go extinct.

Thus, the population is guaranteed not to go extinct for all $a > 0$.

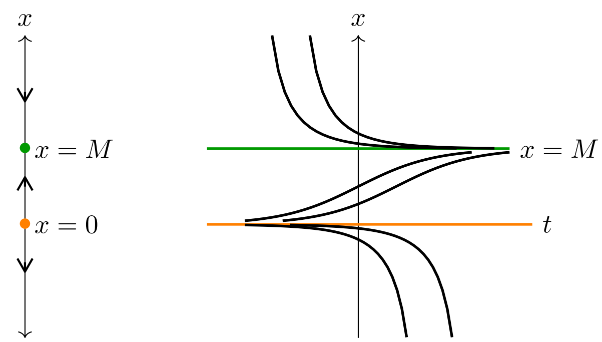
No value of $a > 0$ guarantees against doomsday (x blowing up).

Problem 12.29. *For the following DE, find the critical points, draw the phase line, sketch some integral curves, 'explain' the model.*

Logistic population growth: $x' = kx(M - x)$, *where* $k > 0$

Solution: Critical points are when $f(x) = 0$, which in this case is $x = M$ and $x = 0$. To draw a phase line, we see that

$$x' = kx(M-x) \text{ is } \begin{cases} \text{negative} & \text{for } x > M \\ \text{positive} & \text{for } 0 < x < M \\ \text{negative} & \text{for } x < 0 \end{cases}$$



The phase line shows that $x = M$ is a stable critical point, and $x = 0$ is an unstable critical point.

For the integral curves, we see that above $x = M$ the integral curve has negative slope and goes asymptotically to $x = M$, between $x = 0$ and $x = M$ the curve forms a sort of an s-shape, and for $x < 0$ the curve again has negative slope and curves away from $x = 0$.

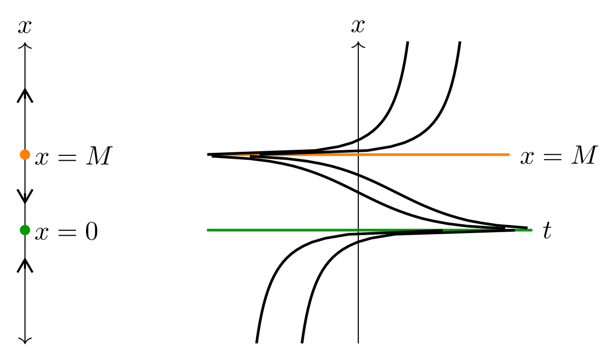
The explanation behind this model is that M is the carrying capacity of the environment: as the population x increases towards M, the growth rate slows, and for populations above $x = M$, animals die off until $x = M$ asymptotically. Populations below zero do not make intuitive sense and we disregard them.

Problem 12.30. *Consider the doomsday-extinction model: $x' = \beta x^2 - \delta x = kx(x - M)$, where $\beta, \delta > 0$. Draw the phase line and sketch some integral curves.*

Solution: Critical points: $x' = kx(x - M) = 0 \Rightarrow x = 0,\, M$.

$$x' = kx(x - M) \text{ is } \begin{cases} \text{positive} & \text{for } x > M \\ \text{negative} & \text{for } 0 < x < M \\ \text{positive} & \text{for } x < 0 \end{cases}$$

This gives the following phase line and solution curves.



This means $x = M$ is an unstable critical point, and $x = 0$ is a stable critical point. If we draw some integral curves, we see that above $x = M$ the line has positive slope and away from $x = M$, between $x = 0$ and $x = M$ the line forms a sort of a backwards s-shape with negative slope, and for $x < 0$ the line again has positive slope and curves towards $x = 0$.

The explanation for why this model is of this form is that we assume births are proportional to x^2, i.e., the probability that two randomly roaming members encounter each other and reproduce and we assume that deathrate is constant. We can see that in our solutions, if we start with $0 < x < M$, eventually everything dies, since the birthrate is not enough to overcome the death rate. If we start with $x > M$, then the population soars to infinity because the births are proportional to x^2, which is very large for large x and the constant death rate cannot keep it in check. We disregard the case $x < 0$ because negative numbers of animals don't make sense.

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3 Extra (not on Quiz 7)

Topic 31 is not on Quiz 7. These problems may help you in reviewing systems of DEs

Topic 31 Applications to physics: mechanical systems

Problem 31.31. **Nonlinear Spring**
The following DE models a nonlinear spring:

$$m\ddot{x} = -kx + cx^3 \quad \begin{cases} \textbf{\textit{hard if}}\ c < 0 & \textit{(cubic term adds to linear force)} \\ \textbf{\textit{soft if}}\ c > 0 & \textit{(cubic term opposes linear force).} \end{cases}$$

(a) *Convert this to a companion system of first-order equations.*

Solution: The companion system is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -kx/m + cx^3/m \end{aligned}$$

(b) *Sketch a phase portrait of the system for both the hard and soft springs. You can use the fact that the linearized centers are also nonlinear centers. (This follows from energy considerations.)*

Solution: Case 1. Hard spring ($c < 0$): One critical point at $(0, 0)$

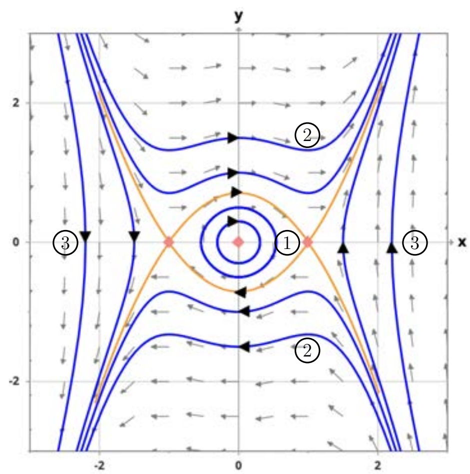
The Jacobian $J(x, y) = \begin{bmatrix} 0 & 1 \\ -k/m + 3cx^2/m & 0 \end{bmatrix}$

$J(0, 0) = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \Rightarrow \lambda = i\sqrt{k/m}$. So we have a linearized center. The problem statement tells us that this is also a nonlinear center.

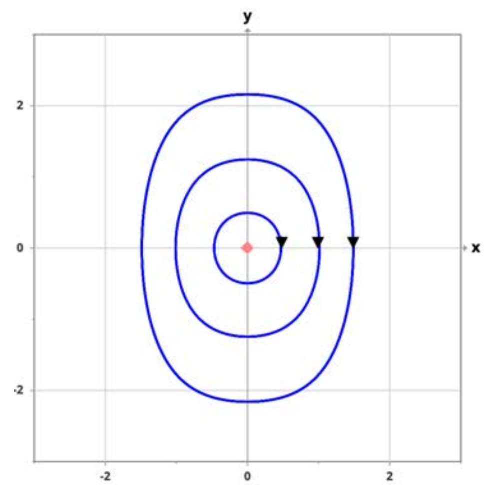
Case 2. Soft spring ($c > 0$): We have the following critical points: $(0, 0)$, $(\pm\sqrt{k/c}, 0)$.

$(0, 0)$: $J(0, 0)$ is the same as for the hard spring. This is a linearized center. The problem statement says it is also a nonlinear center.

$(\pm\sqrt{k/c}, 0)$: $J(\pm\sqrt{k/c}, 0) = \begin{bmatrix} 0 & 1 \\ 2k/m & 0 \end{bmatrix}$ (same for both). Thus we have linearized saddles and, by structural stability, nonlinear saddles. (You should find the eigenvectors to aid in sketching the phase portrait.)



Soft spring: $c > 0$



Hard spring: $c < 0$

(c) *(Challenge! For anyone who is interested. This is not part of the ES.1803 syllabus.) Find equations for the trajectories of the system.*

Solution: We use a standard trick to get trajectories:

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-kx + cx^3}{my}.$$

This is separable: $my\,dy = (-kx + cx^3)\,dy$. Integrating we get

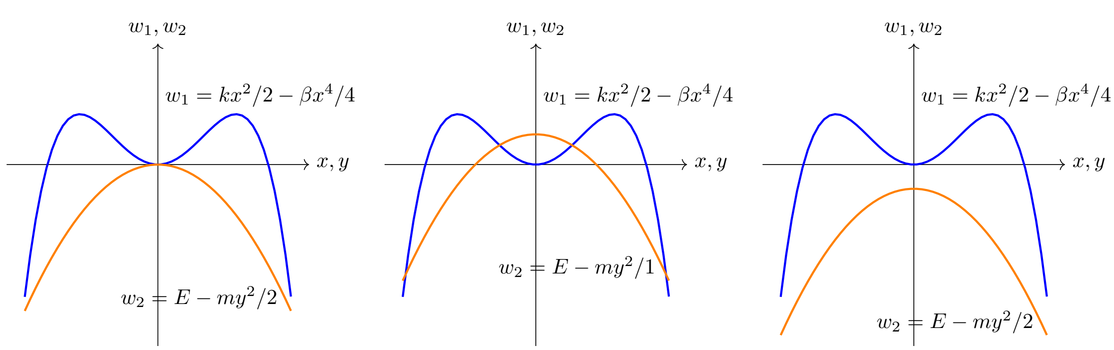
$$\underbrace{\frac{my^2}{2}}_{\text{kinetic energy}} + \underbrace{\frac{kx^2}{2} - \frac{cx^4}{4}}_{\text{potential energy}} = \underbrace{E}_{\text{total energy = constant}}.$$

If $c < 0$ (hard spring), then both energy terms on the right are positive, so x and y must be bounded. Then, for fixed x, there are at most two points on the trajectory. Thus we must have closed trajectories.

If $c > 0$ (soft spring), then, we can define w_1 and w_2 by

$$w_1(x) = \frac{kx^2}{2} - \frac{cx^4}{4}, \quad w_2(y) = E - \frac{my^2}{2}$$

Using $k > 0$, $m > 0$, we have the graphs of w_1, w_2 given below. Using the same graphical ideas as in the proof in the Topic 30 notes that the Volterra predator-prey equation has closed trajectories, this shows the phase plane for the soft spring is as shown above.



Plots of $w_1 = \frac{kx^2}{2} - \frac{cx^4}{4}$, $w_2 = E - y^2$

Different energy levels correspond to different types of trajectories. At the unstable equilibrium we compute $E = \frac{k^2}{4c}$. We have the following correspondence between energy level and trajector (using the labels on the soft-spring phase portrait above):

$E = 0$: Stable equilibrium.

$0 < E < \dfrac{k^2}{4c}$: Trajectories 1.

$E = \dfrac{k^2}{4c}$: Unstable equilibrium, or a trajectory going asymptotically to or from the unstable equilibrium.

$\dfrac{k^2}{4c} < E$: Trajectories 2.

$E < \dfrac{k^2}{4c}$ (including $E < 0$): Trajectories 3

Problem 31.32. *The **damped nonlinear spring** has equation*

$$m\ddot{x} = -kx + cx^3 - b\dot{x}.$$

(a) *Convert it to a system of first-order equations.*

(b) *Sketch a phase portrait for both the hard and soft springs.*

Solution: (a) The system is

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -kx/m + cx^3/m - by/m\end{aligned}$$

(b) Hard spring ($c < 0$): One critical point at $(0, 0)$

$J(0,0) = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \Rightarrow \lambda = \dfrac{-b \pm \sqrt{b^2 - 4km}}{2m}$. So we have 3 possiblities:

(i) underdamped = linearized spiral sink;

(ii) overdamped = linearized nodal sink;

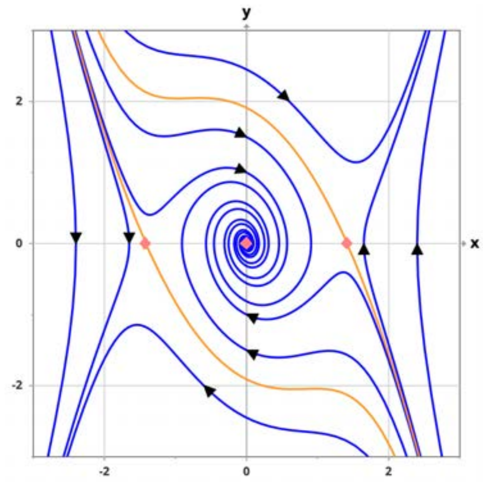
(iii) critically damped = defective sink.

In all cases we have a nonlinear sink. In case (iii), because it's not structurally stable, we would need to do more work to see what type of nonlinear sink we have.

Soft spring ($c > 0$): We have the following critical points: $(0, 0)$, $(\pm\sqrt{k/c}, 0)$.

$(0, 0)$: linearized sink (spiral, nodal or defective), so we have a nonlinear sink.

$(\pm\sqrt{k/c}, 0)$: linearized saddles, so we have nonlinear saddles.



y
x
2
0
-2
-2
0
2