ES.1803 Problem Section Problems for Quiz 7, Spring 2024

This sheet contains problem section problems for **Topics 27-30**. Problem section problems for Topics 10-12 are in a separate file. Those for all other topics are posted with review for previous quizzes.

1 Systems of DEs

Topic 27 Phase portraits of linear 2×2 systems.

Problem 27.1. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 27.2. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 27.3. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 27.4. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 27.5. Draw the trace-determinant diagram. Label all the parts with the type and dynamic stability of the critical point at the origin. Which types represent structurally stable systems?

(b) Give the equation for the parabola in the diagram. Explain where is comes from.

Problem 27.6. Consider the linear system $\mathbf{x}' = A\mathbf{x}$.

(a) Suppose A has $\text{tr}(A) = -2.5$ and $\det(A) = 1$. Locate this system on the trace-determinant diagram. For this system, what is the type of the critical point at the origin?

(b) Compute the eigenvalues of this system and verify your answer in Part (a).

Topic 28 Phase portraits of nonlinear 2×2 systems.

Problem 28.7. **(a)** Sketch the phase portrait for $x' = -x + xy$, $y' = -2y + xy$.

(b) Consider x and y to be the sizes of two interacting populations. Tell a story about the populations.

Problem 28.8. Sketch the phase portrait for $x' = x^2 - y$, $y' = x(1-y)$.

Draw one phase portrait for each possibility for the non-structurally stable critical point.

Problem 28.9. Structural stability using the trace-determinant diagram: Will a non-structurally stable linearized critical point correctly predict the behavior of the nonlinear system at that point?

Problem 28.10. For the following system, draw the phase portrait by linearizing at the critical points.

$$x' = 1 - y^2, \quad y' = x + 2y.$$

Problem 28.11. For the following system, draw the phase portrait by linearizing at the critical points.

$$x' = x - y - x^2 + xy, \quad y' = -y - x^2.$$

Problem 28.12. Consider the system: $x' = x - 2y + 3, \;\; y' = x - y + 2$.

(a) Find the one critical point and linearize at it. For the linearized system, what is the type of the critical point?

(b) In Part (a) you should have found that the linearized system is a center. Since this is not structurally stable, it is not necessarily true that the nonlinear system has a center at the critical point. Nonetheless, in this case, it does turn out to be a nonlinear center. Prove this.

Topic 29: Structural stability

This will be covered in topic 28: nonlinear phase portraits.

Topic 30 Population models

Problem 30.13. Let $x(t)$ be the population of sharks off the coast of Massachusetts and $y(t)$ the population of fish. Assume that the populations satisfy the Volterra predator-prey equations

$$x' = ax - pxy; \qquad y' = -by + qxy, \qquad \text{where } a,\ b,\ p,\ q, \text{ are positive.}$$

Assume time is in years and a and b have units 1/years.

Suppose that, in a few years, warming waters start killing 10% of both the fish and the sharks each year. Show that the shark population will actually increase.

Problem 30.14. Consider the system of equations

$$x'(t) = 39x - 3x^2 - 3xy; \qquad y'(t) = 28y - y^2 - 4xy.$$

The four critical points of this system are (0,0), (13,0), (0,28), (5,8).

(a) Show that the linearized system at (0,0) has eigenvalues 39 and 28. What type of critical point is (0,0)?

(b) Linearize the system at (13,0); find the eigenvalues; give the type of the critical point.

(c) Repeat Part (b) for the critical point (0,28).

(d) Repeat Part (b) for the critical point (5,8).

(e) Sketch a phase portrait of the system. If this models two species, what is the relationship between the species? What happens in the long-run?

Problem 30.15. The system for this equation is

$$x' = 4x - x^2 - xy$$
$$y' = -y + xy$$

(a) This models two populations with a predator-prey relationship. Which variable is the predator population?

(b) What would happen to the predator population in the absence of prey? What about the prey population in the absence of predators?

(c) There are three critical points. Find and classify them

(d) Sketch a phase portrait of this system. What is the relationship between the species? What happens in the long-run?

Problem 30.16. The equations for this system are

$$x' = x^2 - 2x - xy$$
$$y' = y^2 - 4y + xy$$

(a) If this models two populations, what would happen to each of the populations in the absence of the other?

(b) There are four critical points. Find and classify them

(c) Sketch a phase portrait of the system. What is the relationship between the species? What happens in the long-run?

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2 First-order nonlinear

Topic 10: Direction fields, integral curves, existence of solutions

Problem 10.17. Consider $y' = x - y + 1$.

(a) Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as + and negative slope as −. Use this to give a very rough sketch of some solution curves.

(b) Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.

(Note the isocline $y = x$ happens to be a solution—don't expect this to happen usually.)

(c) Can you make a squeezing argument that shows that all solutions go asymptotically to the line $y = x$.

Problem 10.18. Consider $y' = x^2 - y^2$

(a) Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as + and negative slope as −. Use this to give a very rough sketch of some solution curves.

Note: the nullcline consists of two lines.

(b) Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.

(c) Add some integral curves to the plot in Part (b). Include the one with $y(2) = 0$.

(d) Use squeezing to estimate $y(100)$ for the solution with IC $y(2) = 0$.

Problem 10.19. Consider $y' = y(1 - y)$ (Note that there is no x; what does this mean for the shape of your nullclines? Your isoclines?)

(a) Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as + and negative slope as −. Use this to give a very rough sketch of some solution curves.

(b) Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.

Problem 10.20. Consider $y' = y/x$ Note: the line $x = 0$ ($m = \infty$) also separates regions of positive and negative slope.

(a) Sketch the isoclines for $m = 0, \pm 1, \pm 2$. Use it this to give a sketch of some solutions.

(b) This is a rare case where we can solve the DE. Solve the DE and use your solution to draw some integral curves.

Problem 10.21. For $y' = -y/(x^2+y^2)$, sketch the direction field in the upper half-plane. For the solution with initial condition $y(0) = 1$ explain why you know it is decreasing for $x > 0$. Explain why it is always positive for $x > 0$.

Problem 10.22. Consider the DE $y' = \dfrac{1}{x+y}$

Draw a direction field by using about five isoclines; the picture should be square, using the intervals between −4 and 4 on both axes.

Sketch in the integral curves that pass respectively through (0,0), (-1,1), (0,-2). Will these curves cross the line $y = -x - 1$? Explain by using the existence and uniqueness theorem

Problem 10.23. Consider the DE $y' = -xy$.

(a) Draw a direction field using isoclines for $m = 0,\ 1,\ 2,\ -1,\ -2$.

(b) Let $y(x)$ be the solution with initial condition $y(1) = 1.5$. Use fences and funnels to estimate $y(100)$.

Topic 11: Numerical methods

Problem 11.24. For $y' = y^2 - x^2$:

(a) Use Euler's method with $h = 0.5$ to estimate $y(3)$ for the solution with initial condition $y(2) = 0$.

(b) Is the estimate in Part (a) too high or too low?

Problem 11.25. For $\dfrac{dy}{dx} = F(x, y) = y^2 - x^2$.

(a) Use Euler's method to estimate the value at $x = 1.5$ of the solution for which $y(0) = -1$.

Use step size $h = 0.5$. As in the notes, make a table with columns n, x_n, y_n, m, mh.

(b) Is the estimate found in Part (a) likely to be too large or too small?

Topic 12: Automonous DEs and bifurcation diagrams

Problem 12.26. For the following DE, find the critical points, draw the phase line, sketch some integral curves, 'explain' the model.

Temperature: $x' = -k(x - E)$ (E constant ambient temperature).

Problem 12.27. Suppose the following DE models a population $x' = -ax + 1$, which is a constant birth-and-death rate situation modified to include a constant rate of replenishment.

(i) Sketch the bifurcation diagram and list any bifurcation points (these are special values of a).

(ii) The bifurcation points divide the a-axis into intervals. Illustrate one typical case for each interval by giving the phase line diagram. For each of these phase lines, give (rough) sketches of solutions in the tx-plane.

(iii) For what values of a is the population sustainable. What happens for other values of a.

Note the applet 'phase lines' can show this system.

Problem 12.28. Consider the system $x' = x(x - a) + \dfrac{1}{4}$, which is the 'doomsday-vs-extinction' equation with the addition of a constant rate of replenishment.

(a) First consider the equation $x' = x(x-a)$ with $a > 0$. Why is this called the doomsday-vs-extinction population model?

(b) Sketch the bifurcation diagram for $x' = x(x - a) + 1/4$.

(c) Identify the bifurcation points. For what values of a is the population sustainable? Which positive values of a guarantee against extinction? Which positive values of a guarantee against doomsday?

Problem 12.29. For the following DE, find the critical points, draw the phase line, sketch some integral curves, 'explain' the model.

Logistic population growth: $x' = kx(M - x)$, where $k > 0$

Problem 12.30. Consider the doomsday-extinction model: $x' = \beta x^2 - \delta x = kx(x - M)$, where $\beta, \delta > 0$. Draw the phase line and sketch some integral curves.

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3 Extra (not on Quiz 7)

Topic 31 is not on Quiz 7. These problems may help you in reviewing systems of DEs

Topic 31 Applications to physics: mechanical systems

Problem 31.31. Nonlinear Spring
The following DE models a nonlinear spring:

$$m\ddot{x} = -kx + cx^3 \quad \begin{cases} \textbf{hard if } c < 0 & \text{(cubic term adds to linear force)} \\ \textbf{soft if } c > 0 & \text{(cubic term opposes linear force).} \end{cases}$$

(a) Convert this to a companion system of first-order equations.

(b) Sketch a phase portrait of the system for both the hard and soft springs. You can use the fact that the linearized centers are also nonlinear centers. (This follows from energy considerations.)

(c) (Challenge! For anyone who is interested. This is not part of the ES.1803 syllabus.) Find equations for the trajectories of the system.

Problem 31.32. The **damped nonlinear spring** has equation

$$m\ddot{x} = -kx + cx^3 - b\dot{x}.$$

(a) Convert it to a system of first-order equations.

(b) Sketch a phase portrait for both the hard and soft springs.
