Differential Equations Review Sheet 2, Spring 2024

I. Complex Numbers

(a) Euler formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.
(b) Polar form: $a + ib = re^{i\theta}$, $r = \sqrt{a^2+b^2}, \tan\theta = b/a$. (Remember how to draw the polar triangle!)
(c) n^{th} roots of $re^{i\theta}$: $z = r^{\frac{1}{n}} e^{\frac{i\theta}{n} + \frac{i2\pi k}{n}}$, $k = 0, 1, 2, \dots, n-1$.
(d) Complexification: e.g., to solve $P(D)x = F_0 cos(\omega t)$ solve $P(D)z = F_0 e^{i\omega t}$ and then decomplexify: $x = \text{Re}(z)$, or $\int e^{-x}\sin(\omega x)dx = \text{Im}(\int e^{(-1+i\omega)x}dx)$.

II. Linear Constant Coefficient Homogeneous DEs

(a) Linear, Constant Coefficient
1. What does it look like? Well, it's linear, and has constant coefficients.
   We write it as $P(D)x = 0$, where $P(r)$ is a polynomial.
2. How do I solve it?
   Get the characteristic polynomial: replace y by 1, y' by r, y'' by r^2 etc. (This comes from guessing $y = e^{rt}$ as a trial solution.)
3. Solve for the roots of the equation containing r (= characteristic equation).
4. Take roots, r_1, r_2 etc. and arrange as: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \dots$
5. If roots are complex in the form of $a \pm bi$, and you want a real-valued solution, then make them: $y = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt) + \dots$
6. If r is a double root, then e^{rt} and te^{rt} are both homogeneous solutions.

(b) Damping (**2nd-order only:** for $my'' + by' + ky = 0$, m, b, $k > 0$)
1. Underdamping when $b^2 - 4mk < 0$, so roots are complex, solutions oscillate.
2. Overdamping when $b^2 - 4mk > 0$, so roots are real, solutions are exponentials.
3. Critical damping when $b^2 - 4mk = 0$, so roots are repeated,
   solution is $y = c_1 e^{-bt/2m} + c_2 t\, e^{-bt/2m}$.

(c) Stability
1. $y(t) = 0$ is the equilibrium solution.
2. Physics: the system is stable (really asymptotically stable) if the output to the unforced system always goes to the equilibrium as $t \to \infty$.
3. Math: the system is stable if all charactericstic roots have negative real part.
4. Equivalently: the system is stable if all homogeneous solutions to the DE go to 0 as $t \to \infty$.
5. For $my'' + by' + ky = 0$ the system is stable exactly when m, b and k all have the same sign (usually positive).

III. Linear Constant Coefficient Inhomogenous DEs

(a) Preliminaries
1. I'm assuming you can solve the homogenous part, $P(D)y = 0$, already.
2. Inhomogenous CC linear DEs are of the form $P(D)y = f$, with a function $f(t) \neq 0$.
3. The general solution to $P(D)y = f$ is $y = y_p + y_h$.
   (y_p = particular solution, y_h = general homogeneous solution.)

(b) Exponential response formula (ERF)
1. For solving $P(D)x = Be^{at}$.

2. Usual version: $x(t) = \dfrac{Be^{at}}{P(a)}$, if $P(a) \neq 0$.
   Note: a is allowed to be complex.
3. Extended version: if $P(a) = 0$, then the solution is $x(t) = \dfrac{Bte^{at}}{P'(a)}$, if $P'(a) \neq 0$.
4. How do you prove the ERF?
   - Try the solution ce^{at}. After substitution you find this works with $c = B/P(a)$.

(c) Sinusoidal response formula (SRF)

1. For solving $P(D)x = B\cos\omega t$.
2. Usual version: $x(t) = \dfrac{B\cos(\omega t - \phi(\omega))}{|p(i\omega)|}$, if $P(i\omega) \neq 0$.
   Here $\phi(\omega) = \text{Arg}(P(i\omega))$. When writing $\phi(\omega)$ using $\tan^{-1}$, don't forget to give the quadrants where $P(i\omega)$ might lie.
3. How do you prove the SRF?
   - Complexify $P(D)x = B\cos(\omega t)$ to $P(D)z = Be^{i\omega t}$. Then use the ERF.
4. Extended version: if $P(i\omega) = 0$, then $x(t) = \dfrac{Bt\cos(wt - \phi(w))}{|P'(i\omega)|}$, where $\phi(\omega) = \text{Arg}(P'(i\omega))$.

(d) Undetermined coefficients

1. For solving $P(D)x =$ a polynomial.
2. Usual version: guess a solution $x(t) =$ a polynomial of the same degree. Then substitute and solve for the coefficients.
3. **Example.** Solve $x'' + 8x' + 7x = 2t$.
   Solution: Try $x = At + B$.
   Substitution gives $7At + (8A + 7B) = 2t$
   Now equate coefficients: $7A = 2$, $8A + 7B = 0$. (So, $A = 2/7$, $B = -16/49$.)
4. Extended version: If the DE doesn't go all the way to x, then multiply the guess by the right power of t
5. **Example.** Solve $x^{(4)} + 8x''' = 2t$.
   Solution: This only goes to x''', so multiply the guess by t^3
   That is, guess $x = At^4 + Bt^3$.

IV. Linear Operators in General

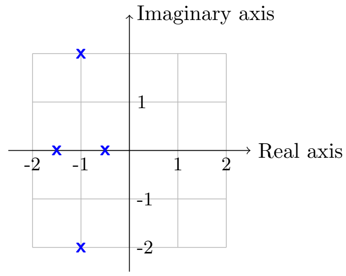
1. An operator T is linear if $T(c_1 f + c_2 g) = c_1 Tf + c_2 Tg$ for all functions f, g and constants c_1, c_2.
2. Our main examples of linear operators are D, $P(D)$.
3. Our main example of a non-linear operator is the squaring operator, $Tf = f^2$.
4. Linearity is almost always easy to check for.

V. Physical Models

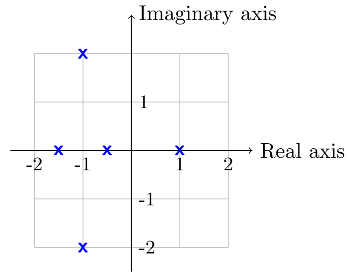
1. Exponential growth and decay: DE is $y' + ky = f(t)$.
2. Spring-mass-dashpot: DE is $my'' + by' + ky = F(t)$,
   where $m =$ mass, $b =$ damping, $k =$ spring constant, $F =$ external (driving) force.
3. LRC circuit: DE is $LI'' + RI' + \frac{1}{C}I = E'$,
   where $L =$ inductance, $R =$ resistance, $C =$ capacitance, $E =$ input voltage.
4. Mixing tanks
   Remember work with amounts not concentrations.
   Rate of change = rate in - rate out.

VI. Pole diagrams

1. For our systems $P(D)x = f$, the pole diagram is drawn in the complex plane. The pole diagram tells us a lot about the homogeneous system.
   - We call the charactistic roots poles.
   - You put an $\times$ at each pole.
   - By counting the poles you can determine the order of the system.
   - If all the poles are in the left half-plane, then the system is stable because all the exponents in the homogeneous solutions have negative real part.
   - If there are complex poles, then the system is oscillatory.
   - For a stable system the exponential rate that the unforced (homogeneous) system returns to equilibrium is determined by the real part of the right-most pole.
2. Examples



4 poles, stable, oscillatory



5 poles, unstable, oscillatory

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