Differential Equations Review Sheet 5, Spring 2024

I. Step and delta functions

A. What are they?

- $u(t)$ is the unit step function: $u(t) = \begin{cases} 0 \text{ for } t < 0 \\ 1 \text{ for } t > 0 \end{cases}$
- $\delta(t)$ is the unit impulse function: $\delta(t) = \begin{cases} 0 \text{ for } t \neq 0 \\ \infty \text{ for } t = 0 \end{cases}$
- $\delta(t)$ is not a regular function, it is called a generalized function

B. What are their properties?

- $u'(t) = \delta(t)$ (generalized derivative)
- $\displaystyle\int_a^b f(t)\delta(t)\,dt = \begin{cases} f(0) \text{ if } 0 \text{ is in } (a,b) \\ 0 \text{ if } 0 \text{ is not in } [a,b] \end{cases}$

  (This requires $f(t)$ is continuous. There are analogous statements for $\delta(t-t_0)$.)
- If $f(t)$ has a jump, then its generalized derivative has a delta function.

  e.g., if $f(t) = \begin{cases} t^2 \text{ for } t < 2 \\ 0 \text{ for } t > 2 \end{cases}$ then $f'(t) = \underbrace{-4\delta(t-2)}_{\text{singular part}} + \underbrace{\begin{cases} 2t \text{ for } t < 2 \\ 0 \text{ for } t > 2 \end{cases}}_{\text{regular part}}$

C. How do they work as input to DEs?

- For an nth order DE an input of $\delta(t)$ causes a jump in the $n-1$ derivitaive.
- e.g., if $mx'' + bx' + kx = \delta(t)$, then x' will have a jump of $1/m$ at $t = 0$.
- **Example.** To solve $mx'' + bx' + kx = \delta(t)$ with $x(0^-) = x'(0^-) = 0$, you break into two cases. Both cases have the same DE:

  $$mx'' + bx' + kx = 0.$$

  The initial conditions change from pre-IC to post-IC.

  For $t < 0$: Pre-IC: $x(0^-) = 0$, $x'(0^-) = 0$.

  For $t > 0$: Post-IC: $x(0^+) = x(0^-) = 0$, $x'(0^+) = x(0^-) + 1/m = 1/m$.

D. Pre and post-initial conditions.

- Pre-initial conditions are values for $x(0^-)$, $x'(0^-)$, $x''(0^-)$, etc. Rest pre-initial conditions (all of the values are 0) are the ones we use the most.
- Post-initial conditions are values for $x(0^+)$, $x'(0^+)$, etc.
- If there is no impulse in the input, then the pre and post-IC are the same.

II. Fourier Series

A. What is it?

- It's used to write a periodic function $f(t)$ as a sum of sines and cosines of multiple frequencies.

B. The general Fourier series for $f(t)$ over $[-L, L]$ with period $= 2L$.

- $\displaystyle f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L}nt\right) + b_n \sin\left(\frac{\pi}{L}nt\right).$
- a_n, b_n are the Fourier coefficients. They are computed using the following integral formulas.
- $a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt.$

- $a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{\pi}{L}nt\right)dt$.
- $b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{\pi}{L}nt\right)dt$.
- The fundamental angular frequency is π/L. All the higher harmonic frequencies are multiples of this.
- The base period is $2L$. Every term in the Fourier series has period $2L$.

C. Simplifying calculations.

1. Even and odd functions.
   - $f(t)$ is even if, $f(t) = f(-t)$; odd if, $f(t) = -f(-t)$.
   - How do I use it?
     - For f even, the Fourier series uses only cosines:

$$f(t) = \frac{a_0}{2} + a_1\cos(\frac{\pi}{L}t)\ldots \text{ where } a_n = \frac{2}{L}\int_0^L f(t)\cos(\frac{\pi}{L}nt)dt.$$

     - For f odd, the Fourier series uses only sines:

$$f(t) = b_1\sin(\frac{\pi}{L}t) + b_2\sin(\frac{\pi}{L}2t)\ldots \text{ where } b_n = \frac{2}{L}\int_0^L f(t)\sin(\frac{\pi}{L}nt)dt.$$

     - Note that these integrals are computed over a half period $[0, L]$.
2. Shifting, scaling, differentiating, integrating
   - If you shift or scale $f(t)$, then you do the same to its Fourier series
   - If you shift or scale t, then you do the same to t in its Fourier series.
   - If you differentiate or integrate $f(t)$, then you do the same to its Fourier series.

D. Fourier cosine and sine series.

1. What are they?
   - Series for functions defined *only* on the interval $[0, L]$.
2. How do I find them?
   - Just like for even and odd functions.
   - Fourier cosine series:

$$f(t) = \frac{a_0}{2} + a_1\cos(\frac{\pi}{L}t)\ldots \text{ where } a_n = \frac{2}{L}\int_0^L f(t)\cos(\frac{\pi}{L}nt)dt.$$

   - Fourier sine series:

$$f(t) = b_1\sin(\frac{\pi}{L}t) + b_2\sin(\frac{\pi}{L}2t)\ldots \text{ where } b_n = \frac{2}{L}\int_0^L f(t)\sin(\frac{\pi}{L}nt)dt.$$

   - Note that these integrals are computed over $[0, L]$.

III. Solving Ordinary DEs with Fourier Series

A. Using Fourier Series to solve inhomogeneous ODEs

- For $P(D)x = f(t)$, where $f(t)$ is periodic:
  1. Express $f(t)$ as a Fourier series.
  2. Break the DE into individual terms.
  3. For each piece: solve using the sinusoidal response formula or compexification.

4. Sum the pieces to get the periodic solution, $x_{sp}(t)$.
5. For undamped second-order, handle cases with pure resonance separately. (That is, cases where $P(i\omega) = 0$.)
6. Often one term dominates the periodic response. That is, often one term is at or near a resonant frequency and the others have a much smaller response.

End of review sheet 5