# Differential Equations Review Sheet 6, Spring 2024

**Also see the review sheet for quiz 5 for more Fourier review.**

## I. Solving Ordinary DEs with Fourier Series

A. Using Fourier Series to solve inhomogeneous ODEs

- For $P(D)x = f(t)$, where $f(t)$ is periodic:
  1. Express $f(t)$ as a Fourier series.
  2. Break the DE into individual terms.
  3. For each piece: solve using the sinusoidal response formula or compexification.
  4. Sum the pieces to get the periodic solution, $x_{sp}(t)$.
  5. For undamped second-order, handle cases with pure resonance separately. (That is, cases where $P(i\omega) = 0$.)
  6. Often one term dominates the periodic response. That is, often one term is at or near a resonant frequency and the others have a much smaller response.

## II. Solving PDEs with Fourier Series

A. Using Fourier Series to solve PDEs

1. Heat equation example on $[0, L]$:
   - PDE: $u_t = au_{xx}$
   - BC: $u(0,t) = 0$, $u(L,t) = 0$. (Ice bath boundary conditions)
   - IC: $u(x,0) = f(x)$. (Initial conditions)
   - To solve this example using separation of variables:
     1. Find all separated solutions of the form $u(x,t) = X(x)T(t)$ satisfying the PDE.
     2. PDE: Substitution gives $X'' + \lambda X = 0$ and $T' + \lambda T = 0$ with the *same* $\lambda$.
     3. Break into cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$. There will be lots of separated solutions to the PDE.
     4. Find the modal solutions (separated solutions that also satisfy the BC). This requires some algebra.
     5. In general, the case $\lambda > 0$ has modal solutions for some set of $\lambda$ which we can list and index. The case $\lambda = 0$ sometimes has modal solutions. The case $\lambda < 0$ never has modal solutions.
     6. For ice bath BC, there are modal solutions when $\sqrt{\lambda} = \frac{n\pi}{L}$. There are no modal solutions in the cases $\lambda = 0$ and $\lambda < 0$. Thus, we can list all the modal solutions

        $$u_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right) e^{-an^2\pi^2 t/L^2} \quad \text{for } n = 1, 2, 3, ...$$

     7. Superposition of the modal solutions gives the Fourier series e.g.,

        $$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-an^2\pi^2 t/L^2}$$

     8. Use ICs to get the Fourier coefficients $b_n$.

- For example if $u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L) = f(x)$, then $b_n$ are the Fourier sine coefficients of $f(x)$:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx$$

2. Wave equation example on $[0, L]$
   - PDE: $y_{tt} = a^2 y_{xx}$ ($a$ is the wave speed)
   - BC: $y(0,t) = 0$, $y(L,t) = 0$. (Boundary conditions for clamped ends)
   - IC: $y(x,0) = f(x)$, $y_t(x,0) = g(x)$. (Initial conditions)
   - Solve this using Fourier's separation of variables method. It is similar to the heat equation example.
3. There are many other type of PDEs and BCs that can be solved using separation of variables. For example,
   - BC: $u_x(0,t) = 0$, $u_x(L,t) = 0$ (insulated boundary conditions).
   - PDE: $y_{tt} + by_t = a^2 y_{xx}$ (damped wave equation).
   - etc.
   - The method works as above. You will need to be careful with the boundary conditions.

*End of review sheet 6*