Differential Equations Review Sheet 7, Spring 2024

I. Drawing Trajectories (solution curves) to 2×2 Linear Systems of DEs

A. Preliminaries

1. Trajectories are solution curves in the xy-plane.
2. We only discuss 2×2 constant coefficient systems $\mathbf{x}' = A\mathbf{x}$.
3. I assume you know how to solve a 2x2 system of DEs to attain its general solution.
4. If $\det A \neq 0$, then the origin $\mathbf{0}$ is the only equilibrium, also called a critical point.
5. Solution curves can never cross each other. So trajectories may approach but never actually reach the origin, which is a solution all by itself.
6. Look at the exponent λ in the $e^{\lambda t}$ factor (that is, the eigenvalue). If it's postive, the normal mode trajectory $e^{\lambda t}\mathbf{v}$ travels away from the origin. If it's negative the normal mode trajectory goes toward the origin.
7. Don't forget to put arrows on your phase portrait!

B. Saddle

1. λ_1 is positive, and λ_2 is negative. Most trajectories are 'deflected' by the equilibrium at the origin. They first approach it and then move away as t increases. The equilibrium is dynamically unstable.
2. Draw lines with the same slope as eigenvectors.
3. Draw solution curves asymptotic to the "arms" of the eigenvectors with arrows as determined in A6 above.

C. Nodes

1. λ_1 and λ_2 are both negative: this is a nodal sink. All trajectories approach the equilibrium at the origin as t increases (arrows point to $\mathbf{0}$). The equilibrium is dynamically asymptotically stable.
2. λ_1 and λ_2 are both positive: source node (arrows point away from $\mathbf{0}$).
3. To determine what the curves look like, test with large positive and negative values of t. Near the origin the trajectory is asymptotic to the eigenvector associated with the smaller (in absolute value) λ and far away from the origin the trajectory is parallel to the eigenvector associated with the larger (in absolute value) λ. Be sure to point the arrows in the correct direction.

D. Spiral or Center

1. λ_1 and λ_2 are $a \pm bi$ so the normal modes involve trigonometric functions.
2. Shape of curves are:
   If $a = 0$, (center) an ellipse results
   If $a < 0$, then it's inward pointing (spiral sink).
   If $a > 0$, then it's outward pointing (spiral source).
3. Dynamic Stability:
   Centers are edge cases –some people call them marginally stable.
   Spiral sinks are dynamically asymptotically stable.
   Spiral sources are dynamically unstable.
4. Determine direction of 'spin' by calculating a tangent vector $\mathbf{x}' = A\mathbf{x}$ at an arbitrary test point $\mathbf{x}$. The spiral or center spins in the direction of $\mathbf{x}'$. A shortcut for this is to check the sign of the lower left hand entry of the coefficient matrix.

E. Other Types

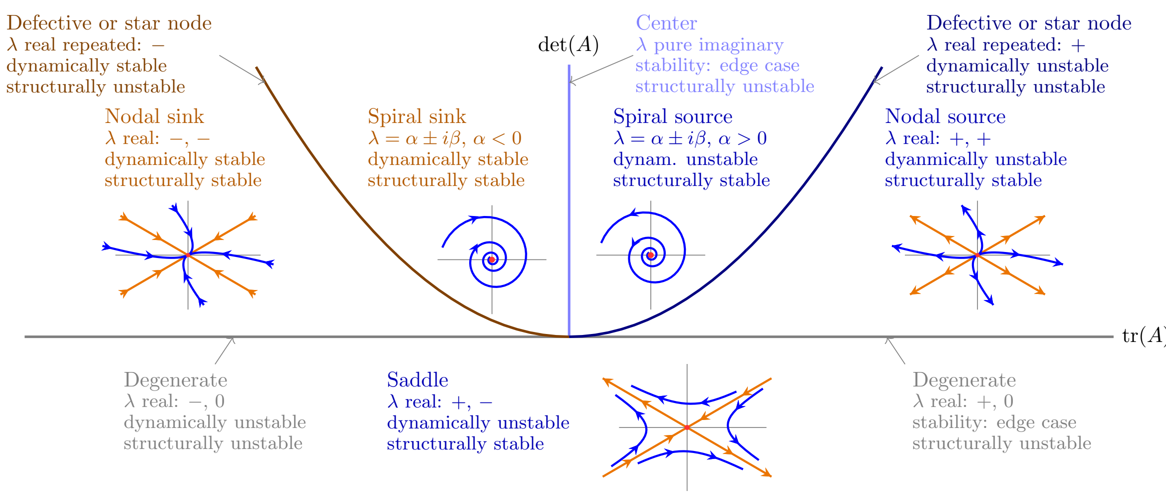
For the other types and a summary see the class notes for Topic 27.

F. Trace-determinant diagram

For a matrix the trace is the sum of the main diagonal entries.
If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\mathrm{tr}(A) = a + d$.
The characteristic equation is

$$\lambda^2 - \mathrm{tr}(A)\,\lambda + \det(A) = 0$$

So the trace and determinant determine the eigenvalues and the type of critical point at the origin.
The trace-determinant diagram summarizes the possible phase portraits.



II. Non-Linear Systems

1. $x' = f(x, y) \quad y' = g(x, y)$.
2. Find the critical points: (x, y) such that $f(x, y) = 0, \quad g(x, y) = 0$. These are the equilibrium solutions.
3. Use the Jacobian $J(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$ to get the coefficient matrix of the linearized system near each critical point.
4. Solve and sketch the (linearized) system of DEs for each critical point.
5. For the structurally stable cases (nodes, saddles and spirals), the trajectories for the non-linear system near the critical point will look essentially the same as those for the linearized system. In these cases, copy the linearized sketches to a little neighborhood of each critical point. For the non-structurally stable cases ('borderline' cases) there are multiple possibilities for the trajectories of the non-linear system.
6. Use your artistic skills to "connect" stray ends of solution curves. Ta-da
7. Interpreting phase portraits.
   Critical points are the same as equilibrium solutions.
   If the system is the companion to a second-order DE, then y is velocity.
   Cycles or spirals in the phase portrait correspond to oscillations of x and y.
   In nature, systems don't remain at unstable equilibria.

End of review sheet 7