ES.1803 Topic 11 Notes

Jeremy Orloff

11 Numerical methods for first-order differential equations

11.1 Goals

1. Be able to compute approximate solutions by hand using Euler's method.
2. Be able to compute the concavity of a solution and say whether Euler's method gives an over or under-estimate,
3. Know some of the ways numerical methods can fail or give misleading results
4. Know the broad outline of how other numerical methods work and understand that many of them are really fancier versions of Euler's method.

11.2 Introduction

In this topic we will look at numerical methods for approximating solutions to differential equations. Just like numerical integration, this allows us to approximate the solution to any first-order DE. It is especially valuable for those equations that we can't solve analytically. Using the computer we can then study as many solutions as we want for a given DE.

11.3 Generalities about numerical methods

The basic framework is that we are given a first-order DE with initial condition

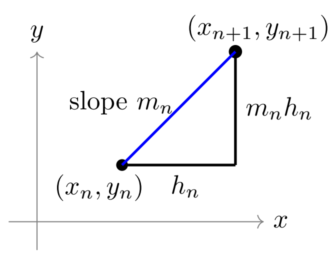
$$y' = f(x, y); \qquad y(x_0) = y_0$$

The goal is to estimate y(x) for other values of x.

The estimate is done by approximating $y(x)$ at a discrete set of points using a series of steps:

Start at (x_0, y_0), step to (x_1, y_1), step to (x_2, y_2), step to (x_3, y_3) ...

Different numerical methods have different ways of computing each step. But they all have the following picture in common.



The triangle shows the step from (x_n, y_n) to (x_{n+1}, y_{n+1}). The horizontal step is h_n. The usual terminology is to call h_n the stepsize at step n. The vertical step is $m_n h_n$, where m_n is the slope at step n.

In the diagram to 'step' from (x_n, y_n) to (x_{n+1}, y_{n+1}) we have

$$x_{n+1} = x_n + h_n; \qquad y_{n+1} = y_n + m_n h_n$$

The job of a numerical method is to specify how to choose h_n and m_n at each step.

11.4 Euler's Method of numerical approximation

Our first method will be Euler's method. Euler's method is very simple to compute and is the only numerical method we will compute by hand. As an aside, it is analogous to using rectangles and Riemann sums to approximate an integral.

Just as in numerical integration, there are fancier numerical methods for solving DEs. These methods require more computation than Euler's and we will leave the computation to computers and existing software packages.

To describe Euler's method we need to say how to choose h_n and m_n for each step.

Euler's method is a fixed stepsize method. This means we fix the stepsize h at the beginning and use it for every step. That is, at each step $h_n = h$.

We know that the slope of the solution curve through (x_0, y_0) is $y' = f(x_0, y_0)$. Euler's method uses this slope to choose m_0, i.e., $m_0 = f(x_0, y_0)$. Likewise, for every subsequent step, Euler's method chooses m_n to be the slope of the direction field at (x_n, y_n), i.e.

$$m_n = f(x_n, y_n)$$

The next example illustrates how to use Euler's method.

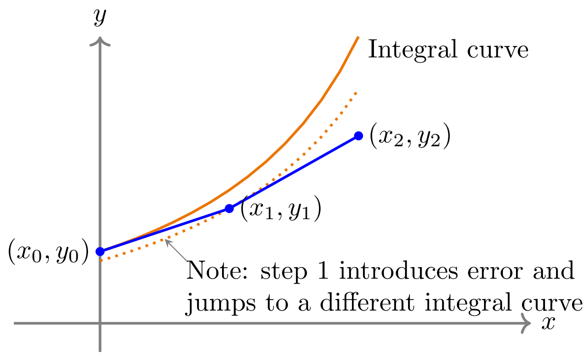
Example 11.1. Numerically solving an IVP using Euler's method. Consider the IVP $y' = x^2 + y^2; \;\; y(0) = -1$. Use Euler's method to estimate $y(1)$.

Solution: We don't know $y(x)$ (and it's hard to find), but we can compute the direction field slope at each point.

Pick a *stepsize*: To keep the computation short, let's take $h = 0.25$. This will take 4 steps to go from $x_0 = 0$ to $x = 1$

Step 0 :	$x_0 = 0$	$y_0 = -1$	$m_0 = 1$	$m_0 h = 0.25$
Step 1 :	$x_1 = 0.25$	$y_1 = -0.75$	$m_1 = 0.63$	$m_1 h = 0.16$
Step 2 :	$x_2 = 0.5$	$y_2 = -0.59$	$m_2 = 0.60$	$m_2 h = 0.15$
Step 3 :	$x_3 = 0.75$	$y_3 = -0.44$	$m_3 = 0.76$	$m_3 h = 0.19$
Step 4 :	$x_4 = 1.00$	$y_4 = -0.25$		

So, $y(1) \approx y_4 \approx -0.25$



Example of Euler's method

In the next example we introduce a simple tabular format for doing and presenting the computation.

Example 11.2. Let $y' = y$; $y(0) = 1$. Estimate $y(1)$

(Note: we know the exact answer, $y = e^x$, $y(1) = 2.718\ldots$)

Let $h = 0.25$, so there are 4 steps from 0 to 1. We organize the calculation in a table:

n	x_n	y_n	$m_n = f(x_n, y_n)$	$m_n h$	actual	error
0	0	1.0	1.0	0.25	1.0	0.0
1	0.25	1.25	1.25	0.31	1.28	0.03
2	0.5	1.56	1.56	0.39	1.65	0.09
3	0.75	1.95	1.95	0.49	2.12	0.17
4	1.0	2.44			2.7183	0.28

Notes:
1. Organize hand calculations like this.
2. Error often accumulates.

Example 11.3. (Example continued.) We now continue the previous example with different stepsizes. In all cases we are trying to estimate $y(1)$.

Stepsize. $h = 1$ (this is just to be a little silly).
With $h = 1$ it takes 1 step to go from 0 to 1.0

n	x_n	y_n	$m_n = f(x_n, y_n)$	$m_n h$	actual	error
0	0	1.0	1	1.0	1.0	0.0
1	1.0	2.0			2.7183	0.72

Stepsize. $h = 0.1$.
With $h = 0.1$ it takes 10 steps to go from 0 to 1.0. Here is the table with some of the numbers left out.

n	x_n	y_n	$m_n = f(x_n, y_n)$	$m_n h$	actual	error
0	0	1	...	...	1	0
1	0.1					
2	0.2	1.21	...	...	1.2214	0.011
3	0.3					
4	0.4	1.4641	...	...	1.4918	0.028
5	0.5					
6	0.6	1.7716	...	...	1.8221	0.05
7	0.7					
8	0.8	2.1436	...	...	2.2255	0.082
9	0.9					
10	1.0	2.5937			2.7183	0.125

Note. The error is smaller when $h = 0.1$ than when $h = 0.25$

Rules of thumb: Using a smaller h is more accurate but requires more computation.

Mild warning. More computation means more risk of roundoff error. In this class, we never make h so small that this is a problem.

11.5 What can go wrong

In this section we'll see that numerical methods can sometimes give misleading results. We hasten to add that numerical methods provide an incredibly powerful tool which is used all the time with great success. But we do need to take some care to avoid certain pitfalls.

We expect that decreasing the stepsize should give a more accurate estimate. The next example shows that we shouldn't simply accept the result, no matter how small the stepsize used.

Example 11.4. Consider the IVP $y' = y^2$; $y(0) = 1$. Use Euler's method to approximate $y(1)$.

Solution: We know the exact solution is $y = \frac{1}{1-x}$, so $y(1) = \infty$. But Euler's method will happily estimate $y(1)$. We do this for several different stepsizes.

Take $h = 0.2$

n	x_n	y_n	$m_n = f(x_n, y_n)$	$m_n h$	actual	error
0	0	1	...	...	1	0
1	0.2	1.2	...	...	1.25	0.05
2	0.4	1.49	...	...	1.67	0.18
3	0.6	1.93	...	...	2.5	0.57
4	0.8	2.68	...	...	5	2.32
5	1.0	4.11			∞	∞

So, $y(1) \approx y_5 = 4.11$.

For decreasing values of h we get the following:
For $h = 0.1$, $y(1) \approx y_{10} = 37.6$.
For $h = 0.05$, $y(1) \approx y_{20} = 91.25$.
For $h = 0.025$, $y(1) \approx y_{40} = 238.21$.

Instead of settling down to a limiting value as we decrease h, the estimate grows. This is a sign that something is wrong with our estimates.

11.5.1 Lesson

You should try smaller and smaller h until the answer settles down. That is, run the estimate with stepsize h. The rerun it with stepsize $h/2$. If the estimates are very close then we have *one* good bit of evidence to accept the estimate as a good approximation. Otherwise, try $h/4$ etc. If the estimate never settles down, then we will have to reject the estimates and use other methods.

The computer doesn't eliminate the need to think!

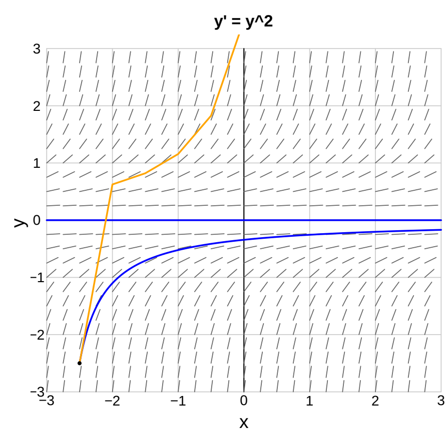
Note. We could make the previous example even more extreme by asking to estimate $y(2)$. The problem is that with the vertical asymptote at $x = 1$ the solution is not even defined at $x = 2$. Nonetheless, for any stepsize h Euler's method will produce an estimate of $y(2)$.

Example 11.5. Stepping across region boundaries. The following shows another risk in using numerical methods. Consider the IVP $y' = y^2$; $y(-2.5) = -2.5$.

The blue curve is the exact solution to the IVP. It goes asymptotically to $y = 0$

The orange curve is the Euler approximation using stepsize $h = 0.5$. It goes off to infinity.

The problem is that the first step in the approximation goes past the separatrix $y = 0$. After that, instead of going asymptotically to 0, the approximation continues to grow.



11.6 Other numerical techniques

All the techniques that we'll look at take steps of the form

$$x_{n+1} = x_n + h;\; y_{n+1} = y_n + m_n h.$$

where m_n is some sort of average slope near (x_n, y_n). The differences between the various methods are in how m_n and possibly h_n is chosen at each step. We'll only touch on this briefly.

Improved Euler (also called RK2). This is a fixed stepsize algorithm, that is we fix the value of h before using it. Here is the algorithm:

1. Start at (x_n, y_n)

2. Compute the slope $k_1 = f(x_n, y_n)$ and take a regular Euler step to a temporary point (x_a, y_a).
$$x_a = x_n + h;\ y_a = y_n + k_1 h.$$
3. Compute the slope at (x_a, y_a): $k_2 = f(x_a, y_a)$.
4. Average the two slopes: $m_n = (k_1 + k_2)/2$.
5. Use m_n as the slope to take the Improved Euler step.
$$x_{n+1} = x_n + h;\ y_{n+1} = y_n + m_n h.$$

Runge-Kutta 4 (RK4). This is also a fixed stepsize algorithm. You can do a web search to get the details. In brief, the algorithm computes 4 different slopes k_1, k_2, k_3, k_4 and then takes a weighted average of these slopes to get m_n. There are different ways to choose the ks and the weights, one common scheme is

$$\begin{aligned} k_1 &= f(x_n, y_n); & k_2 &= f(x_n + h/2, y_n + k_1 h/2); \\ k_3 &= f(x_n + h/2, y_n + k_2 h/2); & k_4 &= f(x_n + h, y_n + k_3 h) \\ m_n &= \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}. \end{aligned}$$

Then as usual,
$$x_{n+1} = x_n + h;\ y_{n+1} = y_n + m_n h.$$

Variable step size methods. There is no reason we have to have a fixed stepsize. It is possible to adjust h at each step. One way to do this is the following:

Suppose we get to (x_n, y_n) with current stepsize h.

1. Take one RK4 step with stepsize h.
2. Repeat with stepsize $h/2$ and $2h$.
3. If the 3 results are very close then change the current stepsize to $2h$ and take the step. If they are not close then change the current stepsize to $h/2$ and take the step.

Thus sometimes the stepsize will get bigger and save computation. When needed to maintain accuracy it will get smaller.

11.7 More technical discussion on error size

(This section is for enrichment only. You will not be asked it on exams.)

For this discussion, we fix a first-order IVP: $y' = f(x, y)$; $y(x_0) = y_0$. We also fix the value x_f and ask to approximate $y(x_f)$.

Euler's method is linear in the error. This means that the error is roughly proportional to h. So, if you halve the stepsize, then you approximately halve the error. Of course, you also double the amount of computation.

Improved Euler is quadratic in the error. This means that the error is roughly proportional to h^2. So, if you halve the stepsize, then the error is approximately quartered.

RK4 is a fourth order method. This means that the error is roughly proportional to h^4. So, if you halve the stepsize, then the error is approximately multiplied by 1/16.

11.8 Second derivative and concavity

If we know $y' = f(x, y)$, then we can find y''. This can be used to determine the concavity of the integral curve and thus, whether the Euler estimate is an over or underestimate.

Example 11.6. Assume $y' = 3xy$ and $y(1) = 2$. Use Euler's method to estimate $y(1.1)$. Is the estimate too high or too low?

Solution: First: $y'(1) = 6$.

Now fix the stepsize $h = 0.1$.

The Euler estimate is $y(1.1) \approx 2 + 0.1 \cdot 6 = 2.6$.

To find the concavity we compute the second derivative. (Note well that y is a function of x.) So,

$$y'' = (3xy)' = 3y + 3xy', \text{ so } y''(1) = y(1) + 3 \cdot y'(1) = 2 + 6 = 8 > 0.$$

We see that y is concave up at $x = 1$ and therefore the Euler estimate is (probably) too low. (Generally speaking, we should be cautious in our statement, because it's possible the graph of y changes concavity between $x = 1$ and $x = 1.1$. In this case, since x, y, y' are all positive, it is clear that $y'' > 0$ for any solution in the first quadrant.)

11.9 Relation to numerical integration

(This section is also just for your enjoyment and enrichment. We won't discuss it in class or on psets or exams.)

Even in 18.01 you were solving (simple) differential equations. A typical 18.01 integration question is to compute $\int_a^b f(x)\,dx$. We can rephrase this as the following initial value problem:

Let $y(x)$ be the solution to the IVP $y' = f(x)$; $y(a) = 0$. What is $y(b)$?

It is clear that this has solution $y(b) = \int_a^b f(x)\,dx$.

Thus for this IVP estimating $y(b)$ with numerical methods amounts to estimating the definite integral using numerical methods. More precisely

Euler's method = numerical integration using left Riemann sums with rectangles.

Improved Euler = numerical integration using the trapezoidal rule.

RK4 = numerical integration using Simpson's rule.

Example 11.7. (Euler's method = left Riemann sum.) For $y' = f(x)$, $y(a) = 0$ estimate $y(b)$ using Euler's method and N steps.

Solution: N steps implies the stepsize is $h = \frac{b-a}{N}$. Thus Euler's method gives

$$y_{n+1} = y_n + f(x_n)\,h.$$

This leads to the following table:

n	x_n	y_n
0	a	0
1	$a+h$	$f(x_0)\,h$
2	$a+2\,h$	$f(x_0)\,h+f(x_1)\,h$
3	$a+3\,h$	$f(x_0)\,h+f(x_1)\,h+f(x_2)\,h$
	...	
N	$a+Nh=b$	$f(x_0)\,h+(f(x_1)+f(x_2)+...+f(x_{N-1}))\ h$

Thus our approximation is $y(b)=\sum_{j=0}^{N-1} f(x_j)\,h$. In 18.01 you might have learned to use Δx instead of h. In either case, the approximation is the left Riemann sum approximating $\displaystyle\int_a^b f(x)\,dx$.