ES.1803 Topic 14 Notes

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14 Row reduction and subspaces

14.1 Goals

1. Be able to put a matrix into row reduced echelon form (RREF) using elementary row operations.
2. Know the definitions of null and column space of a matrix.
3. Be able to use RREF to find bases and describe the null and column spaces of a matrix.
4. Know the definitions of span and independence for vectors.
5. Know the definitions of basis and dimension for a vector space.
6. Know that the column space = $\{\mathbf{b}\}$ for which the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
7. Be able to solve $A\mathbf{x} = \mathbf{b}$ by superpositioning a particular solution and the general homogeneous solution.
8. Be able to describe the geometric effects transforming vectors using matrix multiplication.

14.2 Introduction

Row reduction is a systematic computational method of simplifying a matrix while retaining some of its key properties. This will give us a systematic method of solving systems of linear equations by finding a particular solution and the general homogeneous solution.

The computational goal of row reduction is to simplify the matrix to the so called row reduced echelon form (RREF). Once in this form, we can easily read off some important properties of the original matrix.

Among these properties are the notions of null space and column space, which are two of the fundamental vector subspaces associated to a matrix. In order to discuss these spaces, we will need to learn the general concepts of independence of vectors, and basis and dimension of a vector space.

You will see that we have already seen all of these things, using different terms, in 18.03. We'll illustrate with our standard example: The homogenous equation

$$x'' + 8x' + 7x = 0$$

has two independent solutions $x_1 = e^{-t}$ and $x_2 = e^{-7t}$. Thus the equation has a two dimensional vector space of solutions with basis $\{x_1, x_2\}$. That is, every solution is a linear combination $c_1x_1 + c_2x_2$ of the two basis functions. We say that the space of homogeneous solutions is a two dimensional subspace of the vector space of all functions.

The last section in this topic will introduce the idea that matrix multiplication can be viewed as a transformation or mapping of vectors. At base, this is just the idea that a matrix times a vector is another vector. Looked at geometrically, we will see that matrix multiplication transforms a square to a parallelogram and a circle to an ellipse.

14.3 Row reduction

Row reduction is a computational technique for systematically simplifying a matrix or system of equations. It involves stringing together the elementary row operations listed below. We will see that it is exactly the same as using elimination to solve a system of equations.

We will explain its use through a series of examples.

Elementary row operations

1. Subtract a multiple of one row from another.
2. Scale a row by a non-zero number.
3. Swap two rows.

Example 14.1. Applying row operations to a matrix A.

Start with $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 4 & 12 \end{bmatrix}$. Perform the following row operations.

Subtract 2·Row$_1$ from Row$_2$: $\sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 4 & 12 \end{bmatrix}$.

Subtract 4·Row$_1$ from Row$_3$: $\sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Example 14.2. Use elimination to solve the following system of equations

$$\begin{array}{ccccccc} x & + & & & 2z & = 4 \\ 2x & + & y & + & 7z & = 14 \\ x & + & & & 3z & = 5 \end{array}$$

Solution: We use elimination: subtract 2·Equation$_1$ from Equation$_2$ and at the same time subtract Equation$_1$ from Equation$_3$.

$$\begin{array}{ccccccc} x & + & & & 2z & = 4 \\ & & y & + & 3z & = 6 \\ & & & & z & = 1 \end{array}$$

Now solve the system from the bottom up: $z = 1 \Rightarrow y = 3 \Rightarrow x = 2$.

Let's redo the previous example writing out just the augmented coefficient matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 2 & 1 & 7 & 14 \\ 1 & 0 & 3 & 5 \end{array}\right]$$

Follow the same operations in the example: subtract 2$\cdot$ Row_1 from Row_2 and subtract Row_1 from Row_3.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

This represents the same system of equations and row operations as in the previous example.

14.4 Echelon Form

The final matrix in the previous example is in echelon form. By definition, this means the first non-zero element in each row is farther to the right than the one in the row above. Said differently, below the staircase is all zeros and in the corner of each stair is a nonzero number.

The word echelon seems to have military origins and means a step like arrangement. Here are two examples of matrices in echelon form with the staircase shown. The matrix on the left is the same as the one just above.

$$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & 2 & 4 \\ 0 & \textcircled{1} & 3 & 6 \\ 0 & 0 & \textcircled{1} & 1 \end{array}\right] \qquad \begin{bmatrix} \textcircled{1} & 2 & 2 & 4 & 5 \\ 0 & 0 & \textcircled{1} & 6 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first non-zero element in each row is called a pivot. They are circled in the matrices just above.

14.4.1 Reduced row echelon form (RREF)

A matrix is in reduced row echelon form (RREF) if
1. Each pivot is 1.
2. Each pivot column is all zeros except for the pivot.
3. The rows with all zeros are all at the bottom.

Example 14.3. Use the elementary row operations to put the matrix $\begin{bmatrix} 1 & 2 & 0 & 5 \\ 2 & 4 & 1 & 13 \\ 1 & 2 & 1 & 8 \end{bmatrix}$ in RREF.

Solution: Here are the row operations. R_2 means Row 2 etc. The notation $R_2 = R_2 - 2 \cdot R_1$ means change R_2 by subtracting $2R_1$ from it (like computer code).

$$\begin{bmatrix} 1 & 2 & 0 & 5 \\ 2 & 4 & 1 & 13 \\ 1 & 2 & 1 & 8 \end{bmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 1 & 2 & 1 & 8 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_1} \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} \textcircled{1} & 2 & 0 & 5 \\ 0 & 0 & \textcircled{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in RREF. Again we rewrite it to emphasize the pivots and the echelon.

$$\begin{bmatrix} \textcircled{1} & 2 & 0 & 5 \\ 0 & 0 & \textcircled{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivots of R are circled. The columns with pivots are called pivot columns the other columns are called free columns. We have

$$R \text{ pivot columns: } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad R \text{ free columns: } \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$$

We use these to name the same columns in A:

$$A \text{ pivot columns: } \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad A \text{ free columns: } \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 13 \\ 8 \end{bmatrix}$$

14.4.2 Pivot and free variables

Recall that matrix multiplication results in a linear combination of columns. Using the RREF in Example 14.3 we have

$$\begin{bmatrix} \textcircled{1} & 2 & 0 & 5 \\ 0 & 0 & \textcircled{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} +$$

x_1 and x_3 multiply pivot columns, so they are called pivot variables. x_2 and x_4 multiply free columns, so they are called free variables.

14.5 Column Space of a Matrix

The column space of a matrix is the set of all linear combinations of the columns.

To shorten the phrase 'all linear combinations', we will say it is the span of the columns. In general we have the following definition.

Definition. The span of the vectors $\mathbf{v_1}, \dots, \mathbf{v_n}$ is defined as the set of all linear combinations of the vectors. That is,

$$\text{The span of } \mathbf{v_1}, \dots, \mathbf{v_n} = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \dots + c_n\mathbf{v_n}, \text{ where } c_1, c_2, \dots, c_n \text{ are scalars}\}$$

Important but not hard: make sure you understand why the span of vectors is always a vector space.

Example 14.4. Consider $R = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The column space of R is the set of all vectors of the form

$$x_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + x_2 \begin{bmatrix}2\\0\\0\end{bmatrix} + x_3 \begin{bmatrix}0\\1\\0\end{bmatrix} + x_4 \begin{bmatrix}5\\3\\0\end{bmatrix}$$

Notice that there is some redundancy here: the free columns are clearly linear combinations of the pivot columns. That is

$$\begin{aligned}\text{Column}_2 &= 2 \cdot \text{Column}_1 \\ \text{Column}_4 &= 5 \cdot \text{Column}_1 + 3 \cdot \text{Column}_3\end{aligned}$$

So the free columns are redundant and the column space is given by the span of just the pivot columns:

$$\text{Column space of } R = \text{Col}(R) = \left\{ x_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + x_3 \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$$

Our conclusion is that the pivot columns of R span the column space of R.

Example 14.5. In this example, we'll see that row reduction does not change the relations between the columns. So the pivot columns of A span the column space of A.

The matrix $A = \begin{bmatrix}1 & 2 & 0 & 5\\2 & 4 & 1 & 13\\1 & 2 & 1 & 8\end{bmatrix}$ in Example 14.3 has RREF

$$R = \begin{bmatrix}1 & 2 & 0 & 5\\0 & 0 & 1 & 3\\0 & 0 & 0 & 0\end{bmatrix}.$$

Columns 1 and 3 are the pivot columns of R. Note that for both A and R we have the same relations between the columns, i.e., row reduction did not change these relations:

$$\begin{aligned}\text{Column}_2 &= 2 \cdot \text{Column}_1 \\ \text{Column}_4 &= 5 \cdot \text{Column}_1 + 3 \cdot \text{Column}_3\end{aligned}$$

Therefore, just as with R, the pivot columns in A span its column space. That is,

$$\text{Column space of } A = \text{span}\left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\} = \left\{ c_1 \begin{bmatrix}1\\2\\1\end{bmatrix} + c_2 \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$$

14.6 Rank, basis, dimension, independence

This section is going to throw a lot of vocabulary at you. You need to practice it to make it second nature. You should try to see how most of the new words capture ideas we have been using since the beginning of 18.03

First up is the notion of independence. In Examples 14.4 and 14.5 we saw that the free columns were linear combinations of the pivot columns. This meant they were redundant and not needed to generate the column space. We describe this by saying that the free columns are dependent on the pivot columns.

After getting rid of the free columns it is clear that we need all the pivot columns to make the column space. We describe this by saying that the pivot columns are an independent set of vectors.

The formal definition of independence is the following:
Independence. We say that the vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ are independent if none of them can be written as a linear combination of the others

Example 14.6. (a) Show that vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}$ are not independent.

(b) Show that vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ are independent.

Solution: (a) Note that

$$\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}4\\5\\6\end{bmatrix} + \begin{bmatrix}7\\8\\9\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

This shows that any one of these 3 vectors is a linear combination of the other two. For example

$$\begin{bmatrix}1\\2\\3\end{bmatrix} = 2\begin{bmatrix}4\\5\\6\end{bmatrix} - \begin{bmatrix}7\\8\\9\end{bmatrix}$$

Thus the three vectors are not independent.

(b) One standard way to show independence is to show that the equation

$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$. For Part (b) this is quite obvious since, summing the left hand side, the equation becomes

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Returning to pivot columns, we have the following vocabulary:

- The pivot columns are independent.
- The pivot columns span the column space.
- We combine independence and span into one word and say the pivot columns are a basis for the column space.

- The number of pivot columns is called the dimension of the column space.
- We also call the number of pivots the rank of the matrix. This is the same as the dimension of the column space. It is also the same as the number of non-zero rows in the reduced row echelon form.

Let's restate all our definitions in mathematical terms.

- **Independence:** The vectors $\mathbf{v_1}, \dots, \mathbf{v_n}$ are independent if none can be written as a linear combination of the others. Equivalently, they are independent if the equation (with unknowns $c_1, c_2, \dots, c_n$)
$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \dots + c_n\mathbf{v_n} = 0$$
has only the trivial solution $c_1 = c_2 = \dots = c_n = 0$.
- **Span:** The set of all linear combinations of $\mathbf{v_1}, \dots, \mathbf{v_n}$ is called the span of these vectors. It is a vector space, i.e., closed under addition and scalar multiplication.
- **Basis:** A basis for a vector space is a set of vectors that is both independent and spans the vector space. That is, it gets you the entire space with no redundancy.
- **Dimension:** The dimension of a vector space is the number of vectors in a basis of the space.

Example 14.7.
(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ clearly span $\mathbf{R}^2$. Since they are also independent, they form a basis of $\mathbf{R}^2$. This particular basis is called the standard basis of $\mathbf{R}^2$.

(b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ span $\mathbf{R}^2$. Since they are not independent, e.g., $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a linear combination of the other two vectors, they do not form a basis of $\mathbf{R}^2$.

(c) Since $\mathbf{R}^2$ has a basis with two vectors it has dimension 2.

(d) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ also form a basis of $\mathbf{R}^2$.

To see this we must show that the two vectors are independent and span $\mathbf{R}^2$. It is clear that they are not multiples of each other so they are independent. To see they span $\mathbf{R}^2$ we need to show that any vector $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ can be written as a linear combination of our two vectors. That is, we must always be able solve
$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
for x_1 and x_2. This is just a matrix equation (linear combination of columns)
$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
We can easily solve this by elimination or using the matrix inverse. so we have verified that $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basis of $\mathbf{R}^2$.

14.7 The meaning of the column space

Consider the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

An important problem is to find those vectors $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ for which this equation has a solution. To answer this, remember that matrix multiplication gives a linear combination of the columns. That is, the above matrix equation can be written as

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

We see that the solution to our problem is that the equation has a solution precisely when $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is in the column space of the coefficient matrix.

14.8 Null Space

The null space of a matrix A is the set of all solutions to the homogeneous equation

$$A\mathbf{x} = \mathbf{0}$$

This is exactly the same as what we have often called the homogeneous solution. Mathematicians also use the term kernel as a synonym for null space.

Note. If A has n columns then the null space is a subspace of $\mathbf{R}^n$.

Example 14.8. Find the null space of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 2 \end{bmatrix}$$

Solution: The answer will take a lot of space to display all the vectors and matrices. However, you will see when you work problems on your own that this type of problem does not take a long time to work out.

Also, to make a point, we use the augmented matrix and solve $A\mathbf{x} = \mathbf{0}$ by bringing the augmented matrix to reduced row echelon form.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 2 & 1 & 0 & 2 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & -3 & -2 & -4 & 0 \end{array}\right] \xrightarrow{R_2 = -\frac{1}{3} \cdot R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 2/3 & 4/3 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{cccc|c} \textcircled{1} & 0 & -1/3 & 1/3 & 0 \\ 0 & \textcircled{1} & 2/3 & 4/3 & 0 \end{array}\right]$$

The pivots are circled. The first two columns are pivot columns and the last two are free columns. This last augmented matrix represents a system of equations

$$\begin{bmatrix} 1 & 0 & -1/3 & 1/3 \\ 0 & 1 & 2/3 & 4/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1}$$

We will finish finding the null space by writing these equations out explicitly. (Below, we'll show a more efficient way of presenting the computation.)

Written out as a system of equations, Equation 1 is

$$\begin{aligned} x_1 \qquad - \frac{1}{3}x_3 + \frac{1}{3}x_4 &= 0 \\ x_2 + \frac{2}{3}x_3 + \frac{4}{3}x_4 &= 0 \end{aligned} \tag{2}$$

We can solve for the pivot variables x_1, x_2 in terms of the free variables x_3, x_4:

$$\begin{aligned} x_1 &= \frac{1}{3}x_3 - \frac{1}{3}x_4 \\ x_2 &= -\frac{2}{3}x_3 - \frac{4}{3}x_4 \end{aligned}$$

These equations make it clear that we can set the free variables, x_3, x_4, to any values we choose, i.e., they can be set freely. Once they are set, the pivot variables, x_1, x_2, are fully determined.

So let's set the free variables: $x_3 = a$, $x_4 = b$. With these choices the solution to our equations is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}a - \frac{1}{3}b \\ -\frac{2}{3}a - \frac{4}{3}b \\ a \\ b \end{bmatrix}.$$

This can be written naturally as a linear combination

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = a \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1/3 \\ -4/3 \\ 0 \\ 1 \end{bmatrix},$$

This shows that Null(A) is 2 dimensional with a basis

$$\left\{ \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ -4/3 \\ 0 \\ 1 \end{bmatrix} \right\} \tag{3}$$

Notice that the first basis vector has $x_3 = 1$, $x_4 = 0$. Likewise, the second has $x_3 = 0$, $x_4 = 1$.

In the calculation we just did, x_1, x_2 are pivot variables and x_3, x_4 are free variables. They are called free variables because we could choose their values freely. After that, the pivot variables' values were determined by the Equations 2.

Now, we will show a somewhat more efficient way to present this. The key is to view matrix multiplication as a linear combination of the columns

$$\begin{bmatrix} 1 & 0 & -1/3 & 1/3 \\ 0 & 1 & 2/3 & 4/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1/3 \\ 2/3 \end{bmatrix} + x_4 \begin{bmatrix} 1/3 \\ 4/3 \end{bmatrix} = \mathbf{0}$$

We rewrite this by putting the variables below the columns they multiply:

$$\begin{array}{cccc} \left[\begin{array}{c} 1 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 1 \end{array} & \begin{array}{c} -1/3 \\ 2/3 \end{array} & \left.\begin{array}{c} 1/3 \\ 4/3 \end{array}\right] \\ x_1 & x_2 & x_3 & x_4 \end{array}$$

Then, we find a basis of the null space as follows.

1. Set one free variable to 1 and the other free variables to 0, i.e., write a 1 below one free column and 0s below the other free columns.

2. By inspection choose the values of the pivot variables that make the linear combination of the columns add to 0. Write these values below the pivot columns.

$$\begin{array}{cccc} \left[\begin{array}{c} 1 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 1 \end{array} & \begin{array}{c} -1/3 \\ 2/3 \end{array} & \left.\begin{array}{c} 1/3 \\ 4/3 \end{array}\right] \\ x_1 & x_2 & x_3 & x_4 \\ 1/3 & -2/3 & 1 & 0 \\ -1/3 & -4/3 & 0 & 1 \end{array}$$

So a basis of Null(A) contains the two vectors

$$\mathbf{v_1} = \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v_2} = \begin{bmatrix} -1/3 \\ -4/3 \\ 0 \\ 1 \end{bmatrix}.$$

Now we get the null space of A (all homogeneous solutions) by taking linear combinations of our two basic solutions.

$$\text{Null}(A) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2}\} = \left\{ c_1 \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1/3 \\ -4/3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Of course, this is the same answer we got before.

Remarks. 1. The dimension of the null space equals the number of free variables.

2. We didn't really need to augment the matrix with a column of zeros, since these zeros never changed.

3. For the equation $A\mathbf{x} = \mathbf{b}$, our general approach will be to find a particular solution and the general homogeneous solution. Then we'll use superposition to get the general solution.

This should be very familiar based on what we did with constant coefficient homogeneous DEs.

4. There are of course many other bases of the null space, this is the one we are lead to by our algorithm.

Example 14.9. Consider the matrix

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

This is in row echelon form. Find its null space two ways.

(i) Using our algorithm of setting each free variable, in turn, to 1, find a basis of Null(R). Write the computation below the matrix.

(ii) Explicitly write out the 3 equations in 6 unknowns and solve them.

Finally, note that they produce exactly the same results and convince yourself that they are really identical methods.

Solution: (i) The algorithm to produce a basis of Null(R) says to set, in turn, each free variable to 1 while setting the others to 0. We start by writing the variables below their respective columns. (This reflects the fact that $R\mathbf{x}$ is a linear combination of the columns of R.) So $R\mathbf{x}$ is represented by

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$
$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{matrix}$$

There are 3 free variables, so the null space has dimension 3. We can compute the basis vectors by first setting the free variables and then computing the pivot variables that make the linear combination 0. Here is the computation:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$
$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & -5 & 1 & 0 & 0 \\ -4 & 0 & -6 & 0 & -7 & 1 \end{matrix}$$

In the first row below the variables, we set $x_2 = 1$, $x_4 = 0$, $x_6 = 0$. Then, by inspection we found the values of x_1, x_3, x_5 that made the linear combination of the columns equal 0. In this case, the 1 times Column 2 had to be canceled by -2 times Column 1.

Likewise, in the second row below the variables, we set $x_2 = 0$, $x_4 = 1$, $x_6 = 0$. Then, by inspection, we saw that the 3 and 5 in Column 4, had to be canceled by -3 times Column 1 plus -5 times Column 3.

The three rows below the variables represent a basis of Null(R):

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -6 \\ 0 \\ -7 \\ 1 \end{bmatrix} \right\} \tag{4}$$

(ii) The matrix equation we want to solve is

$$R\mathbf{x} = 0 \Leftrightarrow \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Writing these out explicitly as a system of equations:

$$\begin{array}{ccccccccccccc} x_1 & + & 2x_2 & & & + & 3x_4 & & & + & 4x_6 & = & 0 \\ & & & & x_3 & + & 5x_4 & + & & + & 6x_6 & = & 0 \\ & & & & & & & & x_5 & + & 7x_6 & = & 0 \end{array}$$

Next, solve for the pivot variables x_1, x_3 and x_5 in terms of the free variables x_2, x_4, x_6:

$$\begin{aligned} x_1 &= -2x_2 - 3x_4 - 4x_6 \\ x_3 &= -5x_4 - 6x_6 \\ x_5 &= -7x_6 \end{aligned}$$

Set the free variables freely: $x_2 = a$, $x_4 = b$, $x_6 = c$. With these choices, the solution to our equation $R\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2a - 3b - 4c \\ a \\ -5b - 6c \\ b \\ -7c \\ c \end{bmatrix}.$$

This can be written naturally as a linear combination

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -4 \\ 0 \\ -6 \\ 0 \\ -7 \\ 1 \end{bmatrix}.$$

These vectors are exactly the same as our basis vectors in (4).

The two methods produce exactly the same basis because both involve setting the free variables freely and then computing the pivot variables. In (i), we started by setting one free variable to 1 and the others to 0 to get a basis vector. In (ii), we first found the general solution. Then, the basis vectors were found by setting one free variable to 1 and the others to 0, e.g., setting $a = 0$, $b = 1$, $c = 0$ gives the second basis vector in (4).

14.8.1 Connecting the RREF and the original matrix

The last piece of the puzzle is to connect the null space and column space of a matrix with those of its reduced row echelon form. Let's look again at Example 14.8 and place A and its RREF one above the other

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 2 \end{bmatrix}$$
$$R = \begin{bmatrix} 1 & 0 & -1/3 & 1/3 \\ 0 & 1 & 2/3 & 4/3 \end{bmatrix}$$

Here are the rules.

1. The null space of A is the same as that of R.

2. The column space of R has a basis given by the pivot columns of R. The corresponding columns in A are a basis for the column space of A.

Rule 1 follows because row reduction of the augmented matrix does not alter the solutions to an equation.

Rule 2 follows because row reduction does not change the relationships between the columns.

14.9 Summary of $A\mathbf{x} = \mathbf{b}$

We are now very good at analyzing the equation

$$A\mathbf{x} = \mathbf{b}$$

1. It has a solution if $\mathbf{b}$ is in the column space of A.
2. We can find a solution by elimination (aka row reduction)
3. The full solution is $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$, where $\mathbf{x_p}$ is any particular solution and $\mathbf{x_h}$ is the general homogeneous solution. That is $\mathbf{x_h}$ is the null space of A.
4. We can use reduced row echelon form (RREF) to find a basis and the dimension of both the null space and the column space.

14.10 Matrix multiplication as a linear transformation

It can be very useful to think of matrix multiplication as a function. We'll also say map or linear transform.

Example 14.10. Let $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 2 \end{bmatrix}$. A is a 2×4 matrix so we can multiply it times a 4-vector and get a 2-vector

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

This is a function from $\mathbf{R}^4$ to $\mathbf{R}^2$. We will write

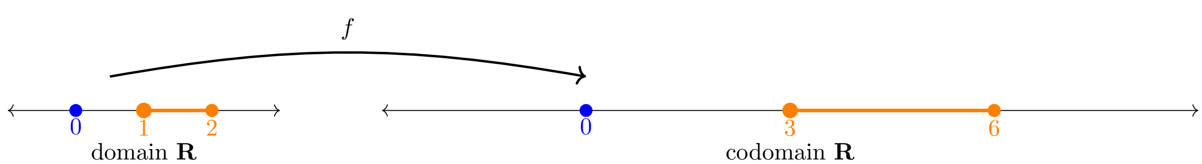
$$\begin{aligned} A : \mathbf{R}^4 &\longrightarrow \mathbf{R}^2 \\ \mathbf{x} &\longmapsto \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 2 \end{bmatrix} \mathbf{x} \end{aligned}$$

and say A maps $\mathbf{R}^4$ to $\mathbf{R}^2$. (More precisely, multiplication by A maps $\mathbf{R}^4$ to $\mathbf{R}^2$.) This is a simple statement, but it is a fruitful way to think about matrix multiplication and will help us understand many things.

14.10.1 Depicting linear transformations with a domain-codomain diagram

Example 14.11. To visualize the function $f(x) = 3x$, we can draw its graph in $\mathbf{R}^2$, i.e., the line $y = 3x$.

But there is another way: Draw the domain and codomain (two copies of the real line) and show where certain features in the domain get mapped (or transformed) to:

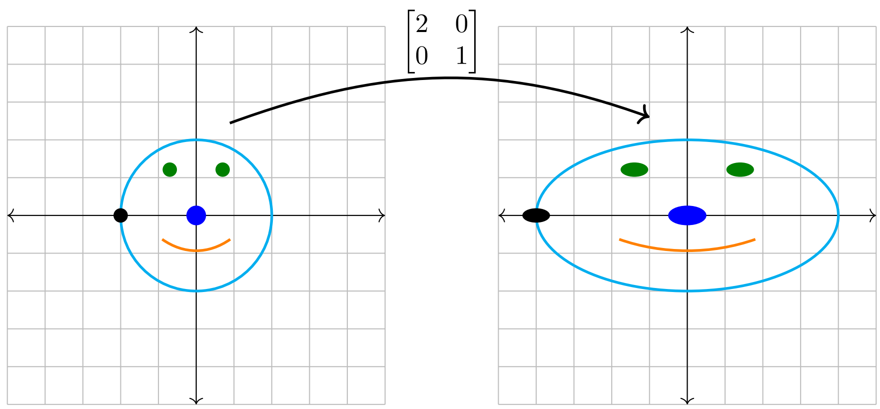


For example, $f(x) = 3x$ maps the point 2 to the point 6 and the interval $[1, 2]$ to the interval $[3, 6]$. The diagram shows how f expands everything by a factor of 3.

Example 14.12. Now consider the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and the associated linear transformation

$$\begin{aligned} \mathbf{f} : \mathbf{R}^2 &\longrightarrow \mathbf{R}^2 \\ \begin{bmatrix} x \\ y \end{bmatrix} &\longmapsto \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}. \end{aligned}$$

Drawing a graph of $\mathbf{f}$ would require 4 dimensions (2 for the input and 2 for the output), so let's draw a domain-codomain diagram instead. How does $\mathbf{f}$ transform Poonen's van Gogh unit smile?

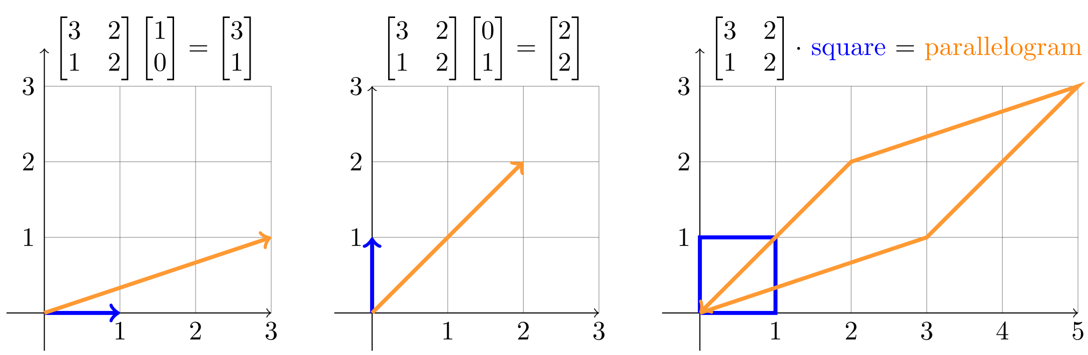


For example, the ear at $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is mapped to $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$. Notice how the linear transformation $\mathbf{f}$ stretches the smiley in the horizontal direction only.

Example 14.13. Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$. For a square matrix, we can save space by putting the domain and codomain in the same plane. For multiplication by A, we have:

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

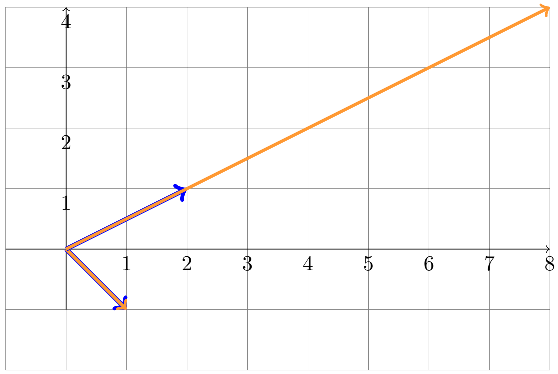
We say $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is mapped to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is mapped to $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$. The figures below display the input vectors in blue and the output vectors in orange. They show that the effect of multiplying a vector by A is to both rotate and scale the input vector. Geometrically the effect of multiplying a square by A is a parallelogram.



Matrix multiplication rotates and scales vectors

As a quick look ahead, we note that most vectors are rotated and scaled, however there are some special vectors that are scaled but not rotated:

$$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$



Special vectors that are not rotated when multiplied by A.

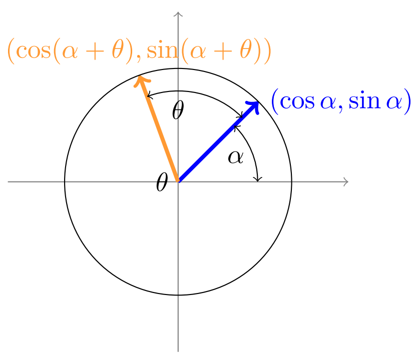
We'll spend a lot time with these special vectors soon. For now let's note the following consequence of linearity: For $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$.

$$A\left(c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 4c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

That's pretty simple!

Example 14.14. Rotation matrices

In this example we'll show that the matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates vectors by an angle θ. To see this we take a unit vector at angle α and see what multiplication by R_θ does to it.



$$R_\theta \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha \\ \sin\theta\cos\alpha + \cos\theta\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{bmatrix} \qquad \text{(trig addition formula!)}$$

The result is a unit vector at angle $\alpha + \theta$, which is what we claimed would happen.

R_θ is called a rotation matrix. We will also use the name orthogonal matrix.

The mathlet `https://mathlets.org/mathlets/matrix-vector/` illustrates matrix multiplication as a mapping of .