ES.1803 Topic 29 Notes

Jeremy Orloff

29 Structural Stability

29.1 Goals

1. Be able to classify a linearized system near a critical point as structurally stable or unstable.

2. For a structurally unstable linearized system, be able to list the possible types of critical point for the nonlinear system.

29.2 Structural stability

Structural stability of the system $\mathbf{x} = A\mathbf{x}$ is about the type of system *not the type of critical point of the system.* Consider the following two scenarios.

Scenario 1. You have an apparatus modeled by a constant coefficient linear system $\mathbf{x}' = A\mathbf{x}$. You are experimentally able to measure the entries of the matrix A to two decimal places of accuracy. You are not suprised when your experiments reveal $A = \begin{bmatrix} 6.00 & 5.00 \\ 1.00 & 2.00 \end{bmatrix}$. So the eigenvalues of your system are 7.00 and 1.00.

You have experimentally determined that the equilibrium at the origin is a nodal source, which is dynamically unstable, i.e., over time trajectories that start near the source move away from it. But we have to take into account the possibility (really, guarantee) of measurement error. Each of your matrix entries might be off by as much as 0.005. Thus the eigenvalues are also only approximately correct.

Nonetheless, with such small errors, the eigenvalues are both guaranteed to be positive and the equilibrium is guaranteed to be a nodal source. We say the system is structurally stable. That is, a small change (also called a perturbation) of the system won't change the type of the equilibrium.

To repeat: the linear system with a nodal source is structurally stable, but has a dynamically unstable equilibrium at the origin.

Scenario 2. You have a known nonlinear system with a critical point at (x_0, y_0). You linearize the system and find that the linearized system has a nodal source with eigenvalues 1 and 7. In this case, the linearized system is an approximation of the nonlinear one. Since, close to the critical point, the approximation error is small, the structural stability of the linearized system tells us that the nonlinear system behaves like a nodal source close to the critical point.

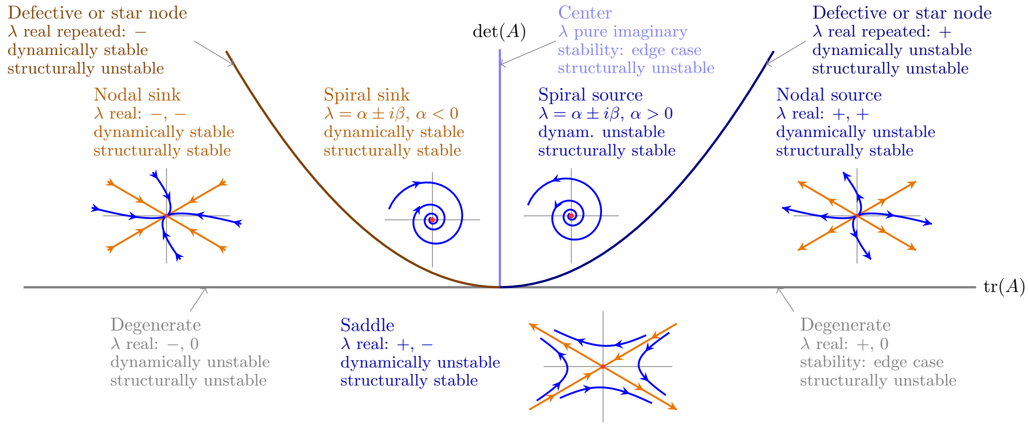
That is, the approximation error changes some fine details of the system, but not the qualitative type of the system. We state this as a theorem

29.3 The open regions in the trace-determinant diagram are structurally stable

Theorem. The linearized system correctly classifies the crititcal point if the linear system is a spiral node, nodal source, nodal sink or saddle.

It may not correctly classify a center, defective node, star node or non-isolated critical point.

That is, it is correct in the open regions of the *trace-determinant* diagram and not definitive on the boundary lines.



The basic idea is that if we 'jiggle' the matrix it won't move very far in the trace-determinant diagram, so the eigenvalues will be of the same type.

29.4 Three examples of a linearized center

The next three examples all have a linearized center at the origin. We will see graphically (and analytically for those who are interested) that a linearized center might be a nonlinear center, spiral source or spiral sink.

Example 29.1. Find the critical points for the system $x' = y - x^2, \quad y' = -x + y^2$. Linearize at each critical point, and say whether the nonlinear system behaves like the linearized system near the point.

Solution: Crititcal points: $y - x^2 = 0$ and $-x + y^2 = 0$.

The first equation implies $y = x^2$. Substitute this in the second equation to get $-x + x^4 = 0$. Thus, $x = 0,\ 1$. So there are two critical points $(0, 0)$ and $(1, 1)$.

Jacobian: $\quad J(x, y) = \begin{bmatrix} -2x & 1 \\ -1 & 2y \end{bmatrix}$.

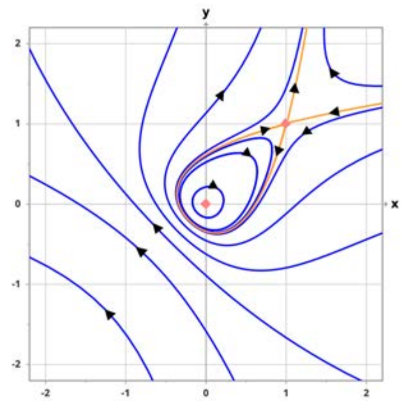
Linearizing:

$J(1, 1) = \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix}$: characteristic equation: $\quad \lambda^2 - 3 = 0 \Rightarrow \lambda = \pm\sqrt{3} \Rightarrow$ linearized system has a saddle.

Since saddles are structurally stable the nonlinear system looks like a saddle at $(1, 1)$.

$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$: eigenvalues $= \pm i \Rightarrow$ a linearized center.

This is **not structurally stable**. Looking at the trace-determinant diagram, a center is on the line between sprial sources and spiral sinks. So the nonlinear system could look like a center, spiral source or spiral sink at $(0,0)$. Using Matlab it appears that (0,0) is a center. (This can be proved analytically.)



The following proof that the critical point is a center is only for those who are interested.

We can show the trajectories near (0,0) are not spirals by exploiting the symmetry of the picture. First note, if $(x(t), y(t)$ is a solution then so is $(y(-t), x(-t)$. That is, the trajectory is symmetric in the line $x = y$. This implies it can't be a spiral. Since the only other choice is that the critical point (0,0) is a center, the trajecories must be closed.

The following two examples show that a linearized center might also be a spiral sink or a spiral source in the nonlinear system.

Example 29.2. $x' = y$, $y' = -x - y^3$.

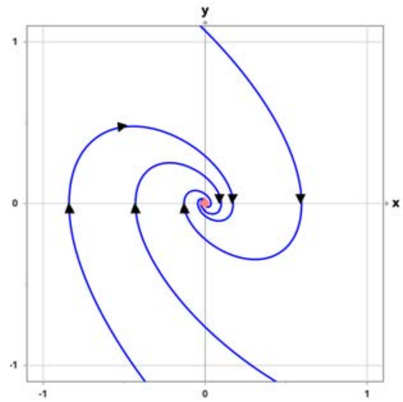
Example 29.3. $x' = y$, $y' = -x + y^3$.

In both examples the only critical point is $(0,0)$.

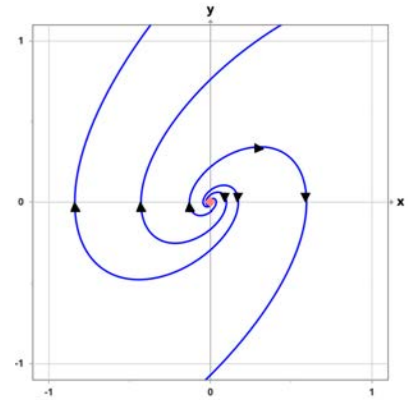
Also, in both examples, $J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. So we have a linearized center at the origin. Again, this is structurally unstable and the nonlinear system could look like a center or a spiral.

In Example 29.2 the critical point turns out to be a spiral sink. In Example 29.3 it is a spiral source. Graphically, using Matlab to plot trajectories, makes this seem reasonable. We can also prove it analytically.

Here are Matlab pictures. (Because the y^3 term causes the spiral to have a lot of turns we 'improved' the pictures by using the power 1.1 instead.)



Spiral in

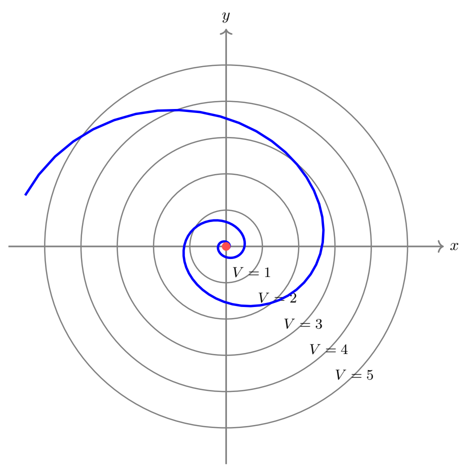


Spiral out

29.4.1 A proof, only for those who are interested.

The proof that these are respectively a spiral source and a spiral sink is based on Lyapunov's second method using the potential function $V(x, y) = x^2 + y^2$.

Consider the system $x' = y, \quad y' = -x - y^3$. If $(x(t), y(t))$ is a solution then $\frac{dV}{dt} = 2x\,x' + 2y\,y' = -2y^4$. Since this is negative or 0 the potential V is decreasing along any trajectory of the system. That is, the trajectory must head towards the origin.



Thus $(0, 0)$ is an asymtotically stable critical point and its type must be a spiral sink.

Likewise, for $x' = y, \quad y' = -x + y^3; \quad \frac{dV}{dt} = y^4 \geq 0$. This implies V is increasing. So the trajectory heads away from origin, i.e. the origin must be a spiral source.