ES.1803 Topic 3 Notes

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3 Input-response models continued

3.1 Goals

1. Be able to use the language of systems and signals.
2. Be familiar with the physical examples in these notes.

3.2 Introduction

In ES.1803 we will use the engineering language of systems and signals. This topic is mostly devoted to learning the vocabulary for this. Our strategy will be to ingrain the words system, signal, input, output (or response) by looking at a series of examples.

It is important to note that these are not mathematical terms and have no formal mathematical definition. Rather, they are engineering terms that will help us organize our thinking when we talk about specific examples. For example, for any given physical model the choice of what to call the input is somewhat arbitrary in the mathematical sense, but usually clear in the engineering sense. What this means in practice is that whenever we need to be mathematically precise, we'll have to say explicitly what we mean by system, input and output. Nonetheless, we'll find the language quite useful. And, in fact, there will be very little confusion when we use these terms.

Another important point is that, in general we will use this language only for **constant coefficient equations** like any of the following:

$$\begin{aligned}
&y' + 3y = cos(t), \\
&y' + ky = q(t), \ \text{ where } k \text{ is a constant}, \\
&my'' + by' + ky = F(t), \ \text{ where } m,\ b,\ k \text{ are constants.}
\end{aligned}$$

3.3 Signals

By **signal** we will simply mean any function of time.

Familiar examples are sound, which is a time varying pressure wave; AM radio signals, where the amplitude of the radio wave varies in time; and FM radio signals, where the frequency of the radio wave varies in time. All of these examples agree with the common definition of signal as something conveying information over time.

In ES.1803 two recurring examples will be the position of a mass oscillating at the end of a spring or the temperature of a body over time. Both of these are clearly functions of time, and, if you think about it, both are conveying information.

3.4 System, input, output (response) by example

We'll now give a series of examples to try to draw out how we use these terms. Remember, even though these choices are natural, they are physical and not mathematical. The key point is that in physical setups we can choose the input and response to be what makes the most sense physically. This needs to be fully specified if there is any chance of confusion.

Example 3.1. Recall the example of my root beer from Topic 2. We have the following model.

$$x' + kx = kE(t), \tag{1}$$

where $x(t)$ is the temperature of the root beer over time and $E(t)$ was the temperature of the environment. Let's describe what we'll choose to be the input, output and system for this setup.

Input signal: I'm interested in how the temperature of my root beer is affected by the environment. With this in mind it is natural to consider $E(t)$ to be the input signal. In general we will shorten this by saying that $E(t)$ is the input.

Output signal or response of the system: The temperature of the root beer changes in response to the input $E(t)$. We're interested in the function $x(t)$, so we call it the response of the system to the input. We usually simplify this by calling it the response or output.

System: The system is the 'mechanism' that that converts the input to the output. In this case the mechanism is the root beer together with the insulation quality of the cooler (measured by k) Mathematically the system is modeled by the differential Equation 1.

Think: What happens to k as the insulation quality of the cooler gets better?

Example 3.2. A spring-mass system. Suppose we have a mass on the end of a spring being pushed by an external force $F(t)$. We'll assume there is no damping, so between Newton and Hooke we have the following DE modeling this system $m\frac{d^2x}{dt^2} = -kx + F(t)$. In ES.1803 we will typically write this with all the x terms on the left-hand side and $F(t)$ on the right:

$$m\frac{d^2x}{dt^2} + kx = F(t), \tag{2}$$

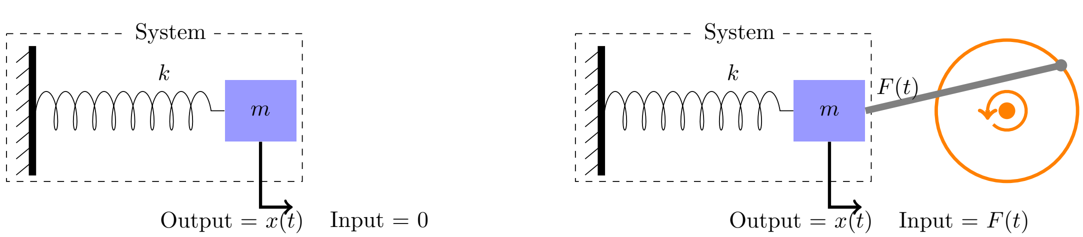
where m=mass, k=spring constant, and $x(t)$=displacement of mass from equilibrium. The following choices seem natural.

System: The spring and mass along with the linkage to the force.

Input: The external force $F(t)$.

Output or response: $x(t)$ the position of the mass over time.

The following figures illustrate this with zero and nonzero input.



Systems with 0 and nonzero input

Example 3.3. Money in the bank. Let $A(t)$ be the amount of money in my retirement account at time t. Suppose also, that interest is paid continuously at the rate r in units of 1/year and that I'm depositing into the account at the rate of $q(t)$ in units of \$/year. While my son was in college $q(t)$ was small. When I retire it will be negative!

Without any deposits or withdrawals $A(t)$ grows exponentially, modeled by $A' = rA$. If we include the deposit rate $q(t)$, we have $A' = rA + q(t)$. We write this with all the A terms on the left and the input $q(t)$ on the right.

$$A' - rA = q(t).$$

Notice that for exponential growth the sign on the rA term is negative. For this situation we will say:

System: Money in the bank earning interest.

Input: The deposit rate $q(t)$.

Output or response: The amount of money in the bank $A(t)$.

3.4.1 Relationship between engineering and mathematical language

We summarize the relationship between the engineering language and the mathematical language as follows:

Physical: system/input produces a response (output)	$\leftrightarrow$	Model: DE has a solution

3.4.2 Mathematical input

This is a math class and we may have the differential equation

$$y' + ky = q(t)$$

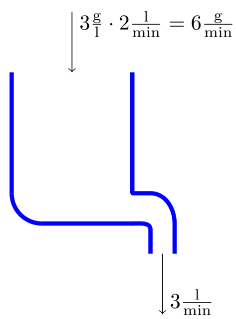
that did not arise from a modeling physical situation. In that case, we will allow ourselves to call the right-hand side $q(t)$ the input and the solution of the DE $y(t)$ the output or response. We will think of $q(t)$ as the mathematical input.

3.5 Worked examples

We'll now work some examples introducing several physical setups that we'll use regularly in this class.

Example 3.4. Mixing tanks. Suppose we have a tank which initially contains 60 liters of pure water. We start adding brine with a concentration of 3 g/liter at the rate of 2 liter/min. While we do this solution leaves the tank at the rate of 3 liter/min. (So the tank will be empty after 60 minutes.)

Assuming instantaneous mixing, find the concentration $C(t)$ of salt in the tank as a function of time.



Mixing tank with inflow and outflow.

Solution: One key lesson in this example is to work with the amount of salt in the tank not the concentration. This is because when you combine solutions the amounts add, but the concentrations do not. At the end we can go back and compute the concentration from the amount and the volume.

Let t be the time in minutes and let $x(t)$ be the amount of salt in the tank at time t in grams. Since 2 liter/min. is entering the tank and 3 liter/min. is leaving the tank, the tank is emptying at the rate of 1 liter/min., i.e., the volume of solution in the tank is given by $V(t) = 60 - t$ liters. The concentration is $C(t) = x(t)/V(t)$.

We know that

$$x'(t) = \text{ rate salt enters the tank} - \text{rate salt leaves the tank.}$$

We can easily compute these rates:

$$\begin{aligned} \text{Rate-in } &= 3\,\frac{\text{g}}{\text{liter}} \cdot 2\,\frac{\text{liter}}{\text{min}} = 6\,\frac{\text{g}}{\text{min}} \\ \text{Rate-out } &= 3\,\frac{\text{liter}}{\text{min}} \cdot \frac{x(t)}{V(t)}\,\frac{\text{g}}{\text{liter}} = \frac{3x}{60-t}\,\frac{\text{g}}{\text{min}}, \end{aligned}$$

Putting this together we have the DE $x'(t) = 6 - \dfrac{3x}{60-t}$. As usual we move all the x terms to the left and get the first-order linear initial value problem

$$x'(t) + \frac{3}{60-t}\,x = 6; \qquad x(0) = 0.$$

This is a first-order linear equation and we can solve it using the variation of parameters formula. We could use the method of finding the general solution and then using the initial condition to find C. Instead, we'll practice the definite integral method. First we find the homogenous solution:

$$x_h(t) = e^{-\int_0^t 3/(60-u)\,du} = e^{3\ln(60-u)|_0^t} = \frac{(60-t)^3}{60^3}.$$

The variation of parameters formula (in definite integral form) is

$$\begin{aligned}
x(t) &= x_h(t)\left(\int_0^t \frac{q(u)}{x_h(u)}\,du + x_0\right) \\
&= \frac{(60-t)^3}{60^3}\left(\int_0^t \frac{6}{(60-u)^3/60^3}\,du + 0\right) \\
&= 6(60-t)^3 \int_0^t \frac{1}{(60-u)^3}\,du \\
&= 6(60-t)^3 \left[\frac{1}{2(60-u)^2}\right]_0^t \\
&= 3(60-t) - \frac{3(60-t)^3}{60^2}.
\end{aligned}$$

To answer the question asked:

$$\boxed{C(t) = \frac{x(t)}{V(t)} = 3 - \frac{3(60-t)^2}{60^2}.}$$

Of course, this model is only valid until $t = 60$ when the tank will be empty.

Example 3.5. A useful format. Consider the exponential growth equation $x' = 3x$ with initial time $t = 5$ and initial condition $x(5) = 2$. A convenient way to write the solution to initial value problem is

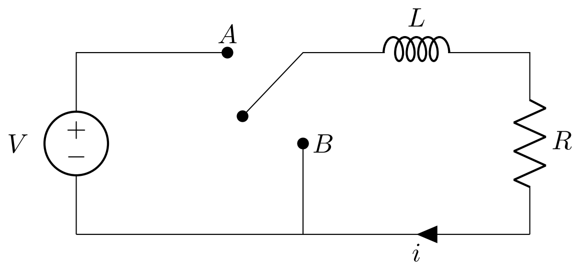
$$x(t) = 2e^{3(t-5)}.$$

This is easy to check by substitution. The point is that since the initial condition is given for $t = 5$ it's easiest to write the solution in terms of $(t - 5)$. This way the coefficient in front of the exponential is just the initial value of x.

Example 3.6. Circuits. (Example of discontinuous input.) An LR circuit is a simple circuit with an inductor L, a resistor R and voltage source V. The differential equation that models the current i is

$$L\frac{di}{dt} + Ri = V.$$

Consider the circuit shown. Assume compatible units and $L = 2$, $R = 4$ and $V = 8$. Also assume that before the switch is closed there is no current in the circuit. At $t = 0$ the switch is moved to position A. Then at $t = 1$ the switch is moved to position B.



Find the current $i(t)$ by writing and solving a differential equation that models this system.

Solution: Each time the switch is moved the input voltage changes. We can write the initial value problem as

$$2i' + 4i = \begin{cases} 8 & \text{for } 0 < t < 1 \\ 0 & \text{for } 1 < t \end{cases}, \qquad \text{with IC } i(t) = 0 \text{ for } t < 0.$$

The format of the input above is called **cases format**. Since the input is given in cases we must solve in cases.

Case (i) For $0 < t < 1$ the DE is: $2i' + 4i = 8$; $i(0) = 0$.
We can solve this using the variation of parameters formula (or by inspection), later we will learn easier techniques: $i(t) = 2 + Ce^{-2t}$. Using the initial condition: $i(0) = 0 = 2 + C$, so $C = -2$. Thus, $\boxed{i(t) = 2 - 2e^{-2t}.}$

To get the initial condition for the next case we find the value of $i(t)$ at the end of this interval: $i(1) = 2 - 2e^{-2}$.

Case (ii) For $1 < t$ the DE is: $2i' + 4i = 0$; $i(1) = 2 - 2e^{-2}$.
Following the format in Example 3.5 we can write the solution to this as
$\boxed{i(t) = i(1)e^{-2(t-1)} = (2 - 2e^{-2})e^{-2(t-1)}.}$

Writing the full solution in **cases format** we have:

$$i(t) = \begin{cases} 0 & \text{for } t < 0 \\ 2 - 2e^{-2t} & \text{for } 0 < t < 1 \\ (2 - 2e^{-2})e^{-2(t-1)} & \text{for } 1 < t \end{cases}$$

Here's a graph of this solution.