ES.1803 Topic 5 Notes

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5 Homogeneous, linear, constant coefficient differential equations

5.1 Goals

1. Be able to solve homogeneous constant coefficient linear differential equations using the method of the characteristic equation. This includes finding the general real-valued solutions when the roots are complex or repeated.
2. Be able to give the reasoning leading to the method of the characteristic equation.
3. Be able to state and prove the principle of superposition for homogeneous linear equations.
4. For a damped harmonic oscillator be able to map the characteristic roots to the type of damping.
5. Be able to create and interpret pole diagrams.

5.2 Introduction

In this topic we will start our study of constant coefficient differential equations. Most of our examples will look at second-order equations, which can be used to model a rich set of physical situations. Second-order equations are fairly simple computationally, yet feature many of the behaviors that higher order equations display.

5.3 Second-order constant coefficient linear differential equations.

The basic second-order constant coefficient linear differential equation can be written as:

$$mx'' + bx' + kx = f(t), \quad \text{where } m, b,\ k \text{ are constants.}$$

The name says it all:

1. Second-order: obvious.
2. Constant coefficient: because the coefficients m, b, k are constant.
3. Linear: derivatives occur by themselves and to the first power. This is the same rule we had for first-order linear, and, just as in that case, we will see that second-order linear equations follow the superposition principle.
4. Note: the ‘input’ $f(t)$ is not necessarily constant.

Reasons to study second-order linear differential equations:

1. There are a lot of second-order physical systems. For example, for moving particles you need the second derivative to capture acceleration.

2. Many higher order systems are built from second-order components.

3. The computations are easy to do by hand and will help us develop our intuition about second-order equations. This computational and intuitive understanding will guide us when we consider higher order equations.

Remark. For second-order systems we will know how they behave and therefore what the solutions to the DEs should look like. For example, a mass oscillating at the end of a spring is a second-order system and we already have a good sense of what happens when we pull on the mass and let it go. So, in some sense, the math is not telling us that much. However, when you couple together 3 springs you have a sixth-order system and our intuition becomes a bit shakier. If you couple even more springs in a two or three dimensional lattice our intuition is shakier still. The success of our second-order models will give us confidence in our higher-order models. And the techniques used to solve second-order equations will carry over to the higher-order case.

5.4 Second-order homogeneous constant coefficient linear differential equations.

For this topic we will focus on the homogeneous equation (H) given just below.

$$mx'' + bx' + kx = 0. \tag{H}$$

We start with an example which pretty well sums up the general technique. Since this is a first example, we will break the solution into small pieces. In later examples we will give solutions that model what we'll expect in your written work.

Example 5.1. (Solving homogeneous constant coefficient DEs: long form solution.) Solve the DE

$$x'' + 8x' + 7x = 0.$$

Solution: 1. Using the method of optimism we guess a solution of the form $x(t) = e^{rt}$. Note that we have left the r unspecified. Our optimistic hope is that the value of r will come out in the algebra.

2. Substitute our guess (trial solution) into the DE:

$$r^2e^{rt} + 8re^{rt} + 7e^{rt} = 0.$$

Divide by e^{rt} (this is okay, it is never 0) to get the **characteristic equation**

$$r^2 + 8r + 7 = 0.$$

3. This has roots: $r = -7, -1$. Therefore, the method of optimism has found two basic solutions:

$$x_1(t) = e^{-7t}, \qquad x_2 = e^{-t}$$

4. Just below, we will discuss the superposition principle, here we will just apply it to get the **general solution** to the DE:

$$\boxed{x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 e^{-7t} + c_2 e^{-t}.}$$

We remind you that the superposition of x_1 and x_2 is also called a linear combination. We will now explain why it works in this case.

5.5 The principle of superposition for linear homogeneous equations

We will state this as a theorem with a proof. The proof is just a small amount of algebra.

Theorem. The superposition principle part 1. If x_1 and x_2 are solutions to (H) then so are all linear combinations $x = c_1 x_1 + c_2 x_2$ where c_1, c_2 are constants.

Proof. As we said, the proof is by algebra. Since we are given a supposed solution, we verify it by substitution, i.e., we plug $x = c_1 x_1 + c_2 x_2$ into (H).

$$\begin{aligned} mx'' + bx' + kx &= m(c_1x_1 + c_2x_2)'' + b(c_1x_1 + c_2x_2)' + k(c_1x_1 + c_2x_2) \\ &= mc_1x_1'' + mc_2x_2'' + bc_1x_1' + bc_2x_2' + kc_1x_1 + kc_2x_2 \\ &= c_1(mx_1'' + bx_1' + kx_1) + c_2(mx_2'' + bx_2' + kx_2) \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &\quad (mx_1'' + bx_1' + kx_1 = 0 \text{ by the assumption that } x_1 \text{ solves (H). Likewise for } x_2.) \\ &= 0. \end{aligned}$$

We have verified that $x = c_1 x_1 + c_2 x_2$ is, in fact, a solution to the homogeneous DE (H).

Superposition = linearity: At this point you should recall the example in Topic 2 where we showed that the nonlinear DE $x' + x^2 = 0$ did not satisfy the superposition principle. It is a general fact that only linear differential equations satisfy the superposition principle.

Example 5.2. (Model solution.) In this example, we suggest a way to give the solutions in your own work. Solve

$$x'' + 4x' + 3x = 0.$$

Solution: Characteristic equation: $r^2 + 4r + 3 = 0$.
Roots: $r = -1, -3$.
Basic solutions: $x_1(t) = e^{-3t}$, $x_2(t) = e^{-t}$.
General solution by superposition: $x(t) = c_1 x_1 + c_2 x_2 = c_1 e^{-3t} + c_2 e^{-t}$.

Note. We call the two solutions x_1, x_2 basic or modal solutions.

Suggestion. For the next week or so every time you use this method remind yourself where each step came from (see the solution to Example 5.1.

Every time we learn a new method we want to test it on our favorite DE.

Example 5.3. Test case: exponential decay. Solve $x' + kx = 0$ using the method of the characteristic equation.

Solution: Characteristic equation (try $x = e^{rt}$): $r + k = 0$.
Roots: $r = -k$.

One solution: $x_1(t) = e^{-kt}$
General solution (by superposition): $x(t) = c_1 x_1 = c_1 e^{-kt}$ (as expected).

In practice, we don't recommend solving this equation with this method. The recommended method is to recognize the DE as the equation of exponential decay and just give the solution.

5.6 Families of solutions

We call $x(t) = c_1 e^{2t} + c_2 e^{-t}$ a two-parameter family of functions. We will often look for subfamilies with special properties.

Example 5.4. (a) Find all the members in the above family that go to 0 as $t \to \infty$.

(b) Find all the members that go to ∞ as $t \to \infty$.

Solution: (a) All the functions $x(t) = c_2 e^{-t}$ (i.e., $c_1 = 0$).

(b) All the functions $x(t) = c_1 e^{2t} + c_2 e^{-t}$, where $c_1 > 0$, c_2 is arbitrary.

5.7 Complex roots

Example 5.5. (Model solution: complex roots) Solve the DE

$$x'' + 2x' + 4x = 0.$$

Solution: 1. Characteristic equation: $r^2 + 2r + 4 = 0$.

2. Roots: $r = (-2 \pm \sqrt{4 - 16})/2 = -1 \pm \sqrt{3}\, i$.

3. Two basic solutions : $x_1(t) = e^{-t}\cos(\sqrt{3}t)$, $x_2(t) = e^{-t}\sin(\sqrt{3}t)$. Here the exponential e^{-t} uses the real part of the roots and the frequency in the sinusoids $\cos(\sqrt{3}t)$, $\sin(\sqrt{3}t)$ comes from the imaginary part of the roots. All of this will be justified below.

4. General *real-valued* solution by superposition:

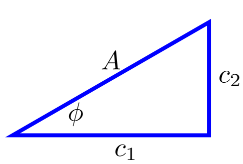
$$x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(\sqrt{3}t).$$

Notes. 1. The damped frequency of oscillation comes from the imaginary part of the roots $\pm\sqrt{3}$.

2. In polar form the solution can be written

$$x(t) = c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(\sqrt{3}t) = Ae^{-t}\cos(\sqrt{3}t - \phi),$$

where A, ϕ, c_1 and c_2 are related by the usual polar triangle with $c_1 = A\cos(\phi)$, $c_2 = A\sin(\phi)$.



Example 5.6. Solve $x'' + 4x = 0$.

Solution: This is the DE for the simple harmonic oscillator a.k.a. a spring-mass system. Using the characteristic equation method:

Characteristic equation: $r^2 + 4 = 0$.
Roots: $r = \pm 2i$.
General real-valued solution: $x = c_1 \cos(2t) + c_2 \sin(2t)$.

Example 5.7. A fifth-order constant coefficient linear homogeneous DE has roots -2, $1 \pm 7i$, $\pm 3i$. What is the general solution?

Solution: $x = c_1 e^{-2t} + c_2 e^t \cos(7t) + c_3 e^t \sin(7t) + c_4 \cos(3t) + c_5 \sin(3t)$.

5.7.1 Justification of the model solution

In Example 5.5, the model solution Steps 1, 2 and 4 are the same as in previous examples with real roots. We need to explain the reasoning behind finding the two basic solutions in step 3:

Amazingly, superposition makes this easy to do. We start with a theorem that tells us how to get real-valued solutions from complex-valued ones.

Theorem. If $z(t)$ is a complex-valued solution to a homogeneous linear DE with real coefficients. Then both the real and imaginary parts of z are also solutions.

Proof. The proof is similar to the proofs of all of our other statements about superposition. Consider the linear homogeneous equation

$$mx'' + bx' + kx = 0 \tag{H}$$

and suppose that $z(t) = x(t) + iy(t)$ is a solution, where $x(t)$ and $y(t)$ are respectively the real and imaginary parts of $z(t)$. We have to show that x and y are also solutions of (H).

By assumption $0 = z'' + bz' + kz$. Replacing z by $x + iy$ we get

$$\begin{aligned} 0 + 0\,i &= m(x+iy)'' + b(x+iy)' + k(x+iy) \\ &= (mx'' + bx' + kx) + i(my'' + by'' + ky). \end{aligned}$$

Since both the real and imaginary parts are 0 we have.

$$mx'' + bx' + kx = 0 \qquad \text{and} \qquad my'' + by' + ky = 0.$$

This says exactly that x and y are solutions to (H).

Now, let's apply this to the situation in Example 5.5.

We saw that there were two characteristic roots $-1 \pm i\sqrt{3}$. These roots give two exponential solutions. Of course, since the roots are complex they are complex exponentials:

$$\begin{aligned} z_1 &= e^{(-1+i\sqrt{3})t} = e^{-t}e^{i\sqrt{3}t} = e^{-t}(\cos(\sqrt{3}t) + i \sin\sqrt{(3t)}) \\ z_2 &= e^{(-1-i\sqrt{3})t} = e^{-t}e^{-i\sqrt{3}t} = e^{-t}(\cos(\sqrt{3}t) - i \sin\sqrt{(3t)}) \end{aligned}$$

Now the theorem above says that both the real and imaginary parts of z_1 and z_2 are also solutions. So we have (nominally) four solutions which we'll label u_1, u_2, v_1, v_2 to avoid overusing the letter x.

$$\begin{aligned} z_1 = u_1 + iv_1 : \quad & u_1(t) = e^{-t}\cos(\sqrt{3}t), \quad v_1(t) = e^{-t}\sin(\sqrt{3}t) \\ z_2 = u_2 + iv_2 : \quad & u_2(t) = e^{-t}\cos(\sqrt{3}t), \quad v_2(t) = -e^{-t}\sin(\sqrt{3}t). \end{aligned}$$

We see that u_1 and u_2 are the same and, except for the minus sign, v_1 and v_2. So we have only two truly different solutions, which is exactly the number we need. These are the basic solutions given in step (3) of Example 5.5 (except that we used the names x_1 and x_2 instead of u_1 and v_1).

5.7.2 Another way to see this

Another way to see that x_1 and x_2 are solutions is to use superposition directly on the two complex exponential solutions. Since z_1 and z_2 are both solutions so are all linear combinations of z_1 and z_2. In particular, x_1 and x_2 are linear combinations of z_1 and z_2 as follows:

$$\begin{aligned} \frac{1}{2}z_1(t) + \frac{1}{2}z_2(t) &= \left(\frac{e^{-t}}{2}\cos(\sqrt{3}t) + i\frac{e^{-t}}{2}\sin(\sqrt{3}t)\right) + \left(\frac{e^{-t}}{2}\cos(\sqrt{3}t) - i\frac{e^{-t}}{2}\sin(\sqrt{3}t)\right) \\ &= e^{-t}\cos(\sqrt{3}t) \\ &= x_1(t). \\ \frac{1}{2i}z_1(t) - \frac{1}{2i}z_2(t) &= \left(\frac{e^{-t}}{2i}\cos(\sqrt{3}t) + i\frac{e^{-t}}{2i}\sin(\sqrt{3}t)\right) - \left(\frac{e^{-t}}{2i}\cos(\sqrt{3}t) - i\frac{e^{-t}}{2i}\sin(\sqrt{3}t)\right) \\ &= e^{-t}\sin(\sqrt{3}t) \\ &= x_2(t). \end{aligned}$$

This shows that x_1 and x_2 are both solutions to the DE.

5.7.3 Complex exponential solutions

We have seen that when the roots of the characteristic equation are complex, we get complex exponentials as solutions. But, with a small amount of algebra, we can write our solutions as linear combinations of real-valued functions. We do this because physically meaningful solutions should have real values. In ES.1803 we won't have much need for the general complex-valued solution, but we record it here for posterity.

The **general complex-valued solution** to the equation in Example 5.5 is

$$z = \tilde{c}_1 z_1 + \tilde{c}_2 z_2 = \tilde{c}_1 e^{(-1+i\sqrt{3})t} + \tilde{c}_2 e^{(-1-i\sqrt{3})t},$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are complex constants. You should be aware that many engineers work directly with these complex solutions and don't bother rewriting them in terms of sines and cosines.

5.8 Repeated roots

When the characteristic equation has repeated roots it will not do to use the same solution multiple times. This is because, for example, $c_1e^{2t} + c_2e^{2t}$ is not really a two-parameter family of solutions, since it can be rephrased as ce^{2t}. For now we will simply assert how to find the other solutions. After we have developed some more algebraic machinery we will be able to explain where they come from.

Example 5.8. A constant coefficient linear homogeneous DE has roots 3, 3, 5, 5, 5, 2. Give the general solution to the DE. What is the order of the DE?

Solution: General solution:

$$x(t) = c_1e^{3t} + c_2te^{3t} + c_3e^{5t} + c_4te^{5t} + c_5t^2e^{5t} + c_6e^{2t}.$$

There are 6 roots so the DE has order 6.

In words: every time a root is repeated we get another solution by adding a factor of t to the previous one.

Example 5.9. A constant coefficient linear homogeneous DE has roots $1 \pm 2i$, $1 \pm 2i$, -3. Give the general real-valued solution to the DE. What is the order of the DE?

Solution: The general real-valued solution is

$$x = c_1e^t\cos(2t) + c_2e^t\sin(2t) + c_3te^t\cos(2t) + c_4te^t\sin(2t) + c_5e^{-3t}.$$

There are 5 roots so the DE has order 5.

5.9 Existence and uniqueness for constant coefficient linear DEs

So far we have rather casually claimed to have found the *general solution* to DEs. Our techniques have guaranteed that these are solutions, but we need a theorem to guarantee that these are all the solutions. There is such a theorem and it is called the existence and uniqueness theorem.

Theorem: Existence and uniqueness. The initial value problem consisting of the DE

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$$

with initial conditions

$$y(t_0) = b_0, \quad y'(t_0) = b_1, \quad \dots, \quad y^{(n-1)}(t_0) = b_{n-1}$$

has a unique solution.

The proof is beyond the scope of this course. The outline of the proof for a general existence and uniqueness theorem is posted with the class notes.

Here is a short explanation for why this theorem guarantees that what we've called the general solution does indeed include every possible solution: The theorem says that there is exactly one solution for each set of initial conditions. Therefore, all we have to show is that our general solution includes a solution matching every possible set of initial conditions.

Matching a set of n initial conditions means solving for the n coefficients c_1, ..., c_n. That is, it means solving a linear system of n algebraic equations in n unknowns. Once we've done more linear algebra we'll be able to show this without difficulty. Right now we'll just look at a representative example.

Example 5.10. Suppose a linear second-order constant coefficients homogeneous DE has characteristic roots 2 and 3. Show that the resulting general solution can match every possible set of initial conditions.

Solution: Our general solution is the two-parameter family

$$x(t) = c_1 e^{2t} + c_2 e^{3t}.$$

Our initial conditions have the form $x(t_0) = b_0$ and $x'(t_0) = b_1$. To match these conditions we have to solve for c_1 and c_2. That is, we have to solve

$$\begin{aligned} x(t_0) &= c_1 e^{2t_0} + c_2 e^{3t} = b_0 \\ x(t_0) &= 2c_1 e^{2t_0} + 3c_2 e^{3t} = b_1 \end{aligned}$$

Writing these equations in matrix form we have

$$\begin{bmatrix} e^{2t_0} & e^{3t_0} \\ 2e^{2t_0} & 3e^{3t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

The coefficient matrix has determinant

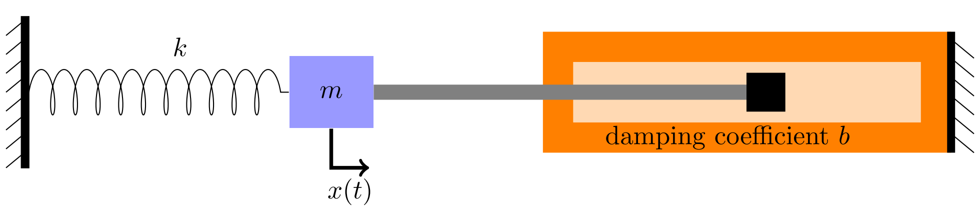
$$\begin{vmatrix} e^{2t_0} & e^{3t_0} \\ 2e^{2t_0} & 3e^{3t_0} \end{vmatrix} = e^{5t_0} \neq 0.$$

Since the determinant is not 0 we know there is a solution giving c_1 and c_2. In fact, we know the solution must be unique.

5.10 Damped harmonic oscillators: the spring-mass-damper

We will use these repeatedly. Please master them.

In ES.1803 one of our main physical examples will be the spring-mass-damper. This is one type of damped harmonic oscillator. (We will encounter others, e.g., an LRC circuit.) In this system we have a mass m attached to a spring with spring constant k. The mass is also attached to a damper that is being dragged through a viscous fluid. The fluid exerts a force on the damper that is proportional to the speed and resists the motion. Let's call the constant in this case the **damping coefficient** b.



Spring-mass-damper with no outside force

For this topic we will assume there is no outside force on the system. So, if $x(t)$ is the displacement of the mass from equilibrium, then Newton's laws tell us

$$mx'' = -kx - bx'.$$

Writing this in our usual fashion, with all the x on the left, we see our standard homogeneous second-order linear constant coefficient DE:

$$mx'' + bx' + kx = 0.$$

A standard notation will be to write $\omega_0 = \sqrt{k/m}$. We'll call ω_0 the **natural frequency** of the system. This term will be explained below.

Simple harmonic oscillator (the undamped spring-mass-dashpot system).

We start with the case of no damping, i.e., $b = 0$. Our equation is then

$$mx'' + kx = 0 \qquad \text{or} \qquad x'' + \omega_0^2 x = 0,$$

where $\omega_0 = \sqrt{k/m}$ = the natural frequency of the oscillator.

Using the characteristic equation method we find:

Characteristic equation: $r^2 + \omega_0^2 = 0$.
Roots: $r = \pm\sqrt{-\omega_0^2} = \pm i\,\omega_0$.
Two solutions: $x_1(t) = \cos(\omega_0 t)$, $x_2(t) = \sin(\omega_0 t)$.
General real-valued solution:

$$x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

We now see why ω_0 is called the natural frequency: it is the angular frequency of the oscillation when the system is undamped and unforced. We will see that damping changes the frequency of oscillation.

Solving the spring-mass-dashpot system: the damped case

Characteristic equation: $mr^2 + br + k = 0$. (Comes from the trial solution $x = e^{rt}$.)

Roots: $r = \dfrac{-b \pm \sqrt{b^2 - 4mk}}{2m}$.

Looking at the formula for the roots we see that there are three cases based on what is under the square root sign. We add a fourth case for when $b = 0$

(i) $b = 0$ **(undamped)**

(ii) $b^2 - 4mk > 0$ (**overdamped;** b large)

(iii) $b^2 - 4mk < 0$ (**underdamped;** b small)

(iv) $b^2 - 4mk = 0$ (**critically damped;** b just right)

We will go through these cases one at a time.

Case (i) (Undamped)

We did this case earlier, The characteristic roots are $\pm\omega_0\, i$. The general real-valued solution is

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

The longterm behavior is periodic (sinusoidal) motion.

Case (ii) (Overdamped: real characteristic roots)

To simplify writing we'll name the expression with the square root. Let $B = \sqrt{|b^2 - 4mk|}$, so the roots are

$$r_1 = \frac{-b+B}{2m} \qquad r_2 = \frac{-b-B}{2m}.$$

First we show that **the roots are real and negative**. This follows because B is the square root of something less than b^2. So, in both $-b+B$ and $-b-B$, the B term is not big enough to change the sign of the $-b$ term. Therefore, r_1 and r_2 must both be negative.

The general real-valued solution to the overdamped system is

$$x(t) = c_1 e^{(-b+B)t/(2m)} + c_2 e^{(-b-B)t/(2m)} = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

The negative exponents imply that in the longterm as t gets large $x(t)$ goes to 0.

The following claim gives an important feature of overdamped systems.

Claim. If an overdamped system starts from rest at a position away from the equilibrium, then it never crosses the equilibrium position.

Since $x = 0$ is the equilibrium position, the claim says that if $x(0) \neq 0$ and $x'(0) = 0$ then the graph of $x(t)$ does not cross the t-axis for $t > 0$.

Proof. The proof involves some picky algebra: We know that the roots r_1 and r_2 are both negative. We also have $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ with initial conditions

$$x(0) = c_1 + c_2 \neq 0, \qquad x'(0) = r_1 c_1 + r_2 c_2 = 0$$

Now we need to show that $x(t) = 0$ never happens for $t > 0$. Let's just do the case $r_1 = -5$ and $r_2 = -2$. The presentation will be simpler and anyone who cares to can redo it for any r_1 and r_2. Using these values of the roots, we have

$$x(t) = c_1 e^{-5t} + c_2 e^{-2t}, \qquad x(0) = c_1 + c_2 \neq 0 \qquad x'(0) = -5c_1 - 2c_2 = 0.$$

The condition $c_1 + c_2 \neq 0$ guarantees that c_1 and c_2 are not both 0. So the other initial condition gives $-c_2/c_1 = 5/2$. Next we'll solve for the times t when $x(t) = 0$.

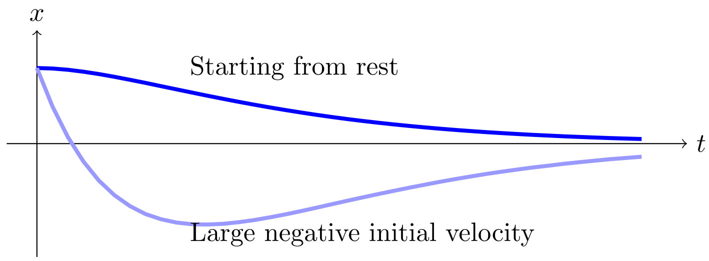
$$x(t) = 0 = c_1 e^{-5t} + c_2 e^{-2t} \quad \text{therefore} \quad -c_2/c_1 = e^{-3t}$$

Combining $-c_2/c_1 = 5/2$ and $-c_2/c_1 = e^{-3t}$, we have $e^{-3t} = 5/2$. Taking the log of both sides we have

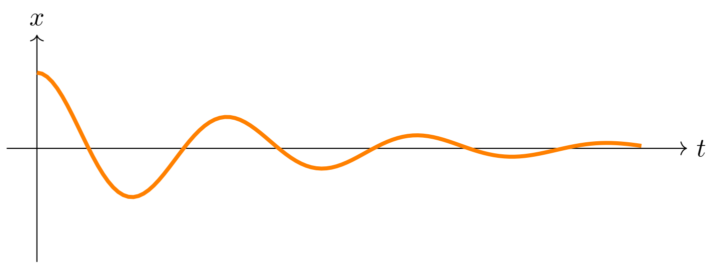
$$-3t = \ln(5/2) > 0, \text{ so, } t < 0.$$

We see that $x(t) = 0$ for exactly one value of t and that value is before $t = 0$. This is exactly what we needed to show!

The proof also showed us that an unforced overdamped harmonic oscillator crosses the equilibrium position at most once.



An overdamped oscillator crosses equilibrium at most once.



An underdamped oscillator crosses equilibrium infinitely many times.

Case (iii) (Underdamping: complex characteristic roots)

Again to simplify the writing we'll name the expression with the square root. Let $B = \sqrt{|b^2 - 4mk|}$. So the characteristic roots are $\frac{-b \pm iB}{2m}$. Just as in Example 5.5, the general real-valued solution is

$$x(t) = e^{-bt/(2m)} \left(c_1 \cos\left(\frac{Bt}{2m}\right) + c_2 \sin\left(\frac{Bt}{2m}\right)\right).$$

Longterm behavior: The negative exponent causes $x(t)$ to go to 0 as t goes to ∞. The sine and cosine causes it to **oscillate** back and forth across the equilibrium.

Case (iv) (Critical damping: repeated real characteristic roots)

In this case the expression under the square root is 0, so we have repeated negative characteristic roots $r = -b/(2m), -b/(2m)$. Thus the general solution to the DE is

$$x(t) = c_1 e^{-bt/(2m)} + c_2 t e^{-bt/(2m)}.$$

Qualitatively the picture looks like the overdamped case. Just as in the overdamped case a critically damped oscillator crosses equilibrium at most once.

5.11 Decay rates

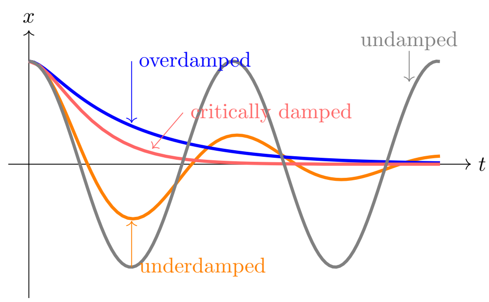
Whether its overdamped, underdamped or critically damped a damped harmonic oscillator goes to 0 as t goes to infinity. We say that $x(t)$ decays to 0. How fast it goes to 0 is its decay rate.

Example 5.11. The rate controlling term. The decay rate of $x(t) = c_1 e^{-3t} + c_2 e^{-5t}$ is the same as that of e^{-3t}. At first glance this might seem surprising because e^{-5t} decays faster than e^{-3t}. But that is exactly the point: the rate of decay is the same as that of the *slowest* term. We might call it the rate controlling term. In this case that is e^{-3t}.

It turns out that critical damping is precisely the level of damping that gives the greatest decay rate. The precise statement is as follows.

Critical damping is optimal. For a fixed mass and spring, i.e., m and k, critical damping is the choice of damping that causes the oscillator to have the greatest decay rate without oscillating.

We will not go through the arithmetic to show this. Anyone interested can ask me about it.



For a fixed m and k critical damping decays the fastest to equilibrium.

5.12 Pole diagrams

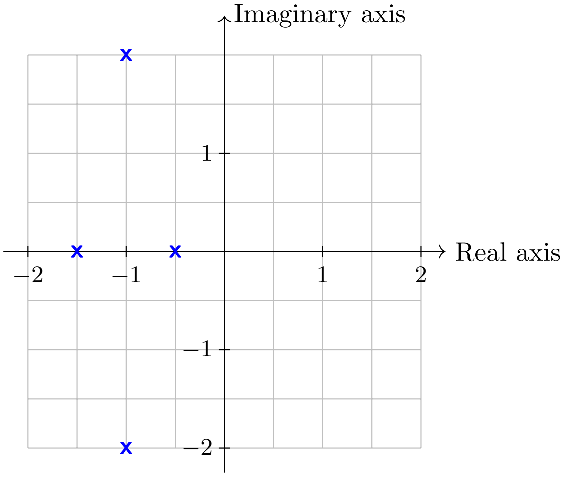
Pole diagrams are a nice way to visualize the characteristic roots of a constant coefficient system $P(D)x = 0$. For these systems the term pole is a synonym for characteristic root. (In general, pole is a mathematical term with a broader meaning.)

The pole diagram is drawn in the complex plane. You construct it by drawing an $\times$ at each pole (characteristic root). It is easy to read off information about the system from the diagram.

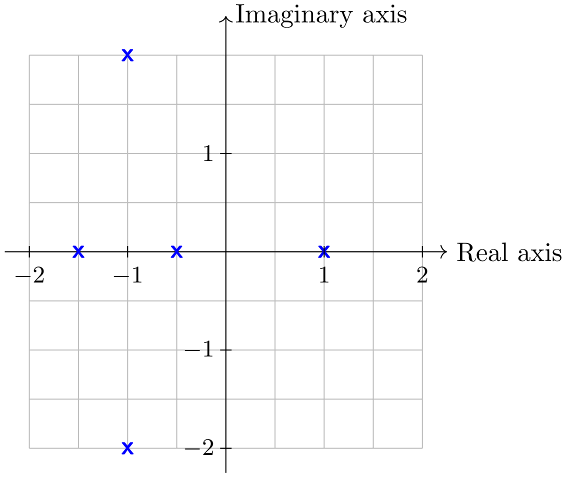
- By counting the poles you can determine the order of the system.
- If all the poles are in the left half-plane then the exponents in the homogeneous solutions all have negative real part. That is, the general homogeneous solution decays to 0, i.e., the system always returns to equilibrium. (We call such a system stable.)
- If there are complex poles then the system is oscillatory.
- For a stable system the exponential rate that the unforced (homogeneous) system returns to equilibrium is determined by the real part of the right-most pole.

Example 5.12. The pole diagram on the left shows 4 poles, all in the left-half plane. Therefore, the system is fourth-order and stable. Since there are complex roots the system is oscillatory. The right-most pole has real part $-1/2$, so the general homogeneous solution decays to 0 like $e^{-t/2}$.

The pole diagram on the right has a pole in the right-half plane at $s = 1$. So the general homogeneous solution grows exponentially, i.e., the system is unstable.



Fourth-order, stable, oscillatory



Fifth-order, unstable, oscillatory

A nice applet showing pole diagrams for second-order systems is the Damped Vibrations applet at https://mathlets.org/mathlets/damped-vibrations/. Set $k = .7$, $m = 1$ and let b vary.