# ES.1803 Topic 8 Notes

Jeremy Orloff

# 8 Applications: stability

## 8.1 Goals

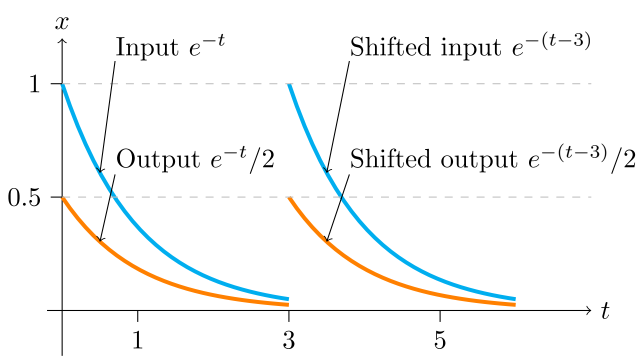
1. Know the meaning of the term 'linear time invariance'.
2. Be able to apply linear time invariance to solve equations with input shifted in time.
3. Know the definitions of mathematical and physical stability
4. Be able to determine if a given 1st, 2nd or 3rd order system is stable.

## 8.2 Time invariance

Constant coefficient differential equations have the property of time invariance. Physically this means that the system responds the same way to an input no matter when the input is started. Mathematically we write this as follows.

**Definition.** Time invariance of a constant coefficient system is the property that if $x_p(t)$ satisfies $P(D)x = f(t)$ then $x_p(t - t_o)$ satisfies $P(D)x = f(t - t_0)$.

**Example 8.1.** We know that $x' + 3x = e^{-t}$ has solution $x_1(t) = e^{-t}/2$. Time invariance says that $x' + 3x = e^{-(t-3)}$ has solution $x_2(t) = x_1(t-3) = e^{-(t-3)}/2$. The figure below illustrates that shifting the input in time simply shifts the output in time.




Physically this has to be the case –an exponential decay system doesn't care what time it gets started.

## 8.3 Mathematical stability

We introduce the idea of stability with an example that shows how negative exponents imply that initial conditions do not affect the long-term behavior of a system.

**Example 8.2.** Consider the DE $x'' + 2x' + 3x = \cos(2t)$

(a) Solve the DE with initial conditions $x(0) = 2, \quad x'(0) = 3$. Describe the long-term behavior of the solution.

(b) Describe the long-term behavior of the solution with initial conditions $x(0) = 1$, $x'(0) = 1$.

**Solution:** (a) First we find the general homogeneous solution.

*Homogeneous solution.*

The characteristic equation is $r^2 + 2r + 3 = 0$. This has roots: $r = -1 \pm \sqrt{2}\,i$.

So, $x_h(t) = c_1 e^{-t} \cos(\sqrt{2})\,t + c_2 e^{-t} \sin(\sqrt{2}\,t)$

*Particular solution.*

Next we find a particular solution using the sinusoidal response formula. For this we need to compute $P(2i)$ and put it in polar form..

$$P(2i) = -4 + 4i + 3 = -1 + 4i = \sqrt{17}e^{i\phi}, \quad \text{where } \boxed{\phi = \text{Arg}(P(2i)) = \tan^{-1}(-4) \text{ in Q2}}.$$

Now the SRF gives $x_p(t) = \dfrac{\cos(2t - \phi)}{\sqrt{17}}$.

*General solution.*

$$x(t) = x_p(t) + x_h(t) = \frac{\cos(2t - \phi)}{\sqrt{17}} + c_1 e^{-t} \cos(\sqrt{2}\,t) + c_2 e^{-t} \sin(\sqrt{2}\,t).$$

Finally, we use the initial conditions to determine the values of $c_1$, $c_2$.

$$\begin{aligned} x(0) &= \cos(-\phi)/\sqrt{17} + c_1 = 2 \;\longrightarrow\; c_1 = 35/17 \\ x'(0) &= -2\sin(-\phi)/\sqrt{17} - c_1 + c_2\sqrt{2} = 3 \;\longrightarrow\; c_2 = 39 * \sqrt{2}/17 \end{aligned}$$

So, $x(t) = \dfrac{\cos(2t - \phi)}{\sqrt{17}} + \dfrac{35}{17} e^{-t} \cos(\sqrt{2}\,t) + \dfrac{39\sqrt{2}}{17} e^{-t} \sin(\sqrt{2}\,t)$.

The question also asks what happens to the system in the long-term, i.e., as $t \to \infty$. Looking at the solution above, we see that the terms with $e^{-t}$ go to 0. This means that, in the long-term, we have

$$x(t) \approx x_p(t) = \frac{\cos(2t - \phi)}{\sqrt{17}}, \quad \text{for large } t.$$

(b) The general solution is the same as in Part (b). Since it has negative exponents, $x_h(t)$ goes to 0 as $t$ goes to infinity. This means that, in the long-term, the solution $x(t)$ behaves exactly like the solution in Part (a), i.e., goes asymptotically to $x_p(t)$.

This is the key point: the values of $c_1$ and $c_2$ will change with the initial conditions, but in the long-term, the terms with $c_1$ and $c_2$ will go to 0, i.e., the initial conditions don't affect the long-term behavior of the system.

This leads to our definition of stability and several equivalent ways of describing it.

**Definition.** Mathematical stability means the long-term behavior doesn't depend (significantly) on initial conditions.

**Linear Systems.** The system $Ly = f$ is stable if the general homogeneous solution $y_h(t) \to 0$ as $t \to \infty$. In this case, $y_h$ is called the transient.

**Linear CC Systems.** The system $P(D)y = f$ is stable if all the characteristic roots have negative real part.

For linear systems stability is determined by the homogeneous solution. That is,

**Stability is about the system not the input.**

**Example 8.3.** $x' + 2x = f(t)$ is stable because $x_h(t) = ce^{-2t} \to 0$.

**Example 8.4.** A constant coefficient system with roots $-2 \pm 3i$, $-3$ is stable.

**Example 8.5.** A constant coefficient system with roots $-2$, $-3$, $4$ is unstable.

**Example 8.6.** $P(D)y = y'' + 8y' + 7y = f(t)$ has characteristic roots -7, -1. These are negative so the system is stable.

**Example 8.7.** $P(D)y = y'' - 6y' + 25y = f$ has characteristic roots $3 \pm 4i$. The real parts of these roots are positive, so the system is not stable.

## 8.4 Stability criteria for linear CC systems

1. Stability $\Leftrightarrow$ for any IC $y_h \to 0$ as $t \to \infty$.

2. Stability $\Leftrightarrow$ all characteristic roots have negative real part.

3. Stability $\Leftrightarrow$ all solutions to the homogeneous equation $P(D)y = 0$ go asymptotically to the homogeneous equilibrium solution $y(t) = 0$.

4. For a first-order system $P(D)y = y' + ky = f(t)$:
   Characteristic root $= -k$. Therefore, stability $\Leftrightarrow$ $k > 0$.

5. For a second-order system $P(D)y = my'' + by' + ky = f(t)$:
   Stability $\Leftrightarrow$ $m$, $b$, $k$ all have the same sign (easy to prove).

6. For a third-order system $P(D)y = y''' + ay'' + by' + cy = f$:
   Stability $\Leftrightarrow$ $a, b, c > 0$ and $ab > c$ (harder to prove).

   This shows that third-order systems with positive coefficients aren't necessarily stable.

   **Example:** An unstable system with positive coefficients
   $$(r + 5)(r - 1 - 100i)(r - 1 + 100i) = r^3 + 3r^2 + 96r + 505.$$

7. The stability criteria for third-order systems is an example of the Routh-Hurwitz stability criteria, which is described below in the last section of this topic.

   **Key point:** This criteria is somewhat complicated, but it allows us to determine stability from the coefficients of a system. That is, it does not require finding the roots!

## 8.5 Physical stability

**Definition.** Physical stability. An unforced physical system with a single equilibrium is called stable if, for any initial conditions, it always returns to the equilibrium.

Later in the course we will expand on the notion of stability for systems with multiple equilibria. The next example shows how physical and mathematical stability are related.

**Example 8.8.** Damped-spring-mass system: Physical stability matches mathematical stability. The equilibrium solution is $x(t) = 0$. The unforced system is modeled by $mx' + bx' + kx = 0$. Since the roots have negative real part, $x(t) \to 0$, no matter what the initial conditions.

Note: The previous section on stability criteria show that second-order physical systems, like springs and LRC circuits are always stable. This is not true of 3rd (and higher) order physical systems. An example is given in the in-class notes for this topic which discuss Maxwell's model of steam engines.

## 8.6 Routh-Hurwitz stability criteria

This section is copied from Section S of the 18.03 Supplementary Notes by Arthur Mattuck. We include it for anyone who is interested. You are not responsible for knowing this in 18.03.

Assume $a_0 > 0$, the constant coefficient, linear system

$$(a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n I)x = f(t)$$

is stable if and only if

*in the matrix below, all of the $n$ principal minors (i.e., the subdeterminants in the upper left corner having sizes respectively $1, 2, \ldots, n$) are greater than 0.*

$$\begin{bmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & a_{2n-4} & \ldots & \ldots & \ldots & a_n \end{bmatrix}$$

*In the matrix, we define $a_k = 0$ if $k > n$. Thus, for example, the last row always has just one non-zero entry, $a_n$.*

The proof of this is some fairly elaborate algebra, which we won't reproduce here.

**Example 8.9.** Apply the Routh-Hurwitz criteria to the system

$$x''' + ax'' + bx' + cx = f(t).$$

**Solution:** The matrix for this system is

$$\begin{bmatrix} a & 1 & 0 \\ c & b & a \\ 0 & 0 & c \end{bmatrix}$$

The three principle minors are

$$|a| = a, \quad \begin{vmatrix} a & 1 \\ c & b \end{vmatrix} = ab - c, \quad \begin{vmatrix} a & 1 & 0 \\ c & b & a \\ 0 & 0 & c \end{vmatrix} = c(ab - c)$$

The Routh-Hurwitz criteria are that all three minors must be positive. That is,

$$a > 0, \quad ab - c > 0, \quad c(ab - c) > 0$$

Since $ab-c > 0$, the condition $c(ab-c) > 0$ implies $c > 0$. Then, since $a$ and $c$ are positive, the condition $ab - c > 0$ implies $b > 0$. Thus we have the criteria stated above:

The system is stable is equivalent to $a$, $b$, $c$ are positive and $ab > c$.

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