Lagrangian and Eulerian representations of fluid flow

Part 1: Kinematics and the equations of motion

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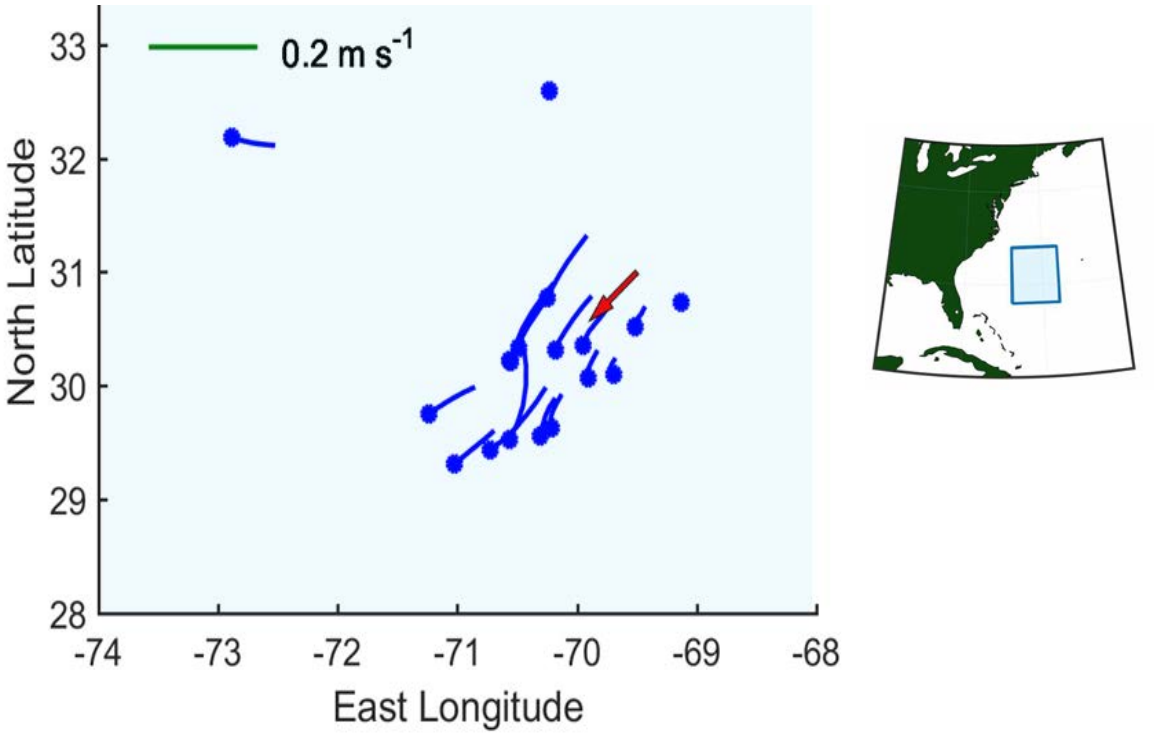

Figure 1: Lagrangian and Eulerian measurements of ocean currents at a depth of 1300 m in the subtropical western North Atlantic. The blue comets are five day-long (Lagrangian) trajectories of neutrally buoyant floats, and the red vector is the (Eulerian) current measured by a fixed current meter and scaled similarly. To see these data in motion visit https://www2.whoi.edu/staff/jprice/wp-content/uploads/sites/199/2022/06/LDE1300.mp4 The goal of this essay is to help you develop an understanding of both the Lagrangian and the Eulerian systems, and especially to appreciate how they may be used side-by-side in the analysis of a fluid flow.

Abstract: This essay introduces the two systems that are widely used to observe and analyze fluid flows. The first system, termed Lagrangian, seeks to observe or calculate the trajectories of specific fluid parcels. Lagrangian methods will sometimes be the most efficient way to sample a fluid flow, and, the physical conservation laws are inherently Lagrangian since they apply to specific fluid volumes. However, the Lagrangian equations of motion applied to a three-dimensional continuum are awkward for many applications, and thus the majority of theory in fluid mechanics has been developed within the second system, termed Eulerian, which seeks to observe or calculate the fluid velocity at fixed positions. The premise of this essay is that an improved understanding and appreciation of both systems will help you build the framework for your continued study of fluid mechanics.

The transformation of the conservation laws from a Lagrangian to an Eulerian system can be envisaged in three steps. (1) The first is dubbed the Fundamental Principle of Kinematics: the fluid velocity at a given time and fixed position (the Eulerian velocity) is equal to the velocity of the fluid parcel (the Lagrangian velocity) that is present at that position at that instant. Thus while we often speak of Lagrangian velocity or Eulerian velocity, it is important to keep in mind that these are merely (but significantly) different ways to represent a given fluid flow. (2) A similar relation holds for time derivatives of fluid properties, say a passive tracer, c: the time rate of change observed on a specific fluid parcel, $Dc/Dt = \partial c/\partial t$ in the Lagrangian system, has a counterpart in the Eulerian system, $Dc/Dt = \partial c/\partial t + \mathbf{V} \cdot \nabla c$, called the material derivative. The material derivative at a given position is equal to the Lagrangian time rate of change of the parcel that is present instantaneously at that position. And lastly, (3) The physical conservation laws apply to extensive quantities, i.e., the mass or the momentum of a specific fluid volume. The time derivative of the integral of a tracer over a moving fluid volume (a Lagrangian quantity) can be transformed into the equivalent Eulerian time derivative for the corresponding intensive quantity (the tracer concentration) by means of the Reynolds Transport Theorem.

In an Eulerian system, transport by the fluid flow is represented by the advection term of the material derivative, $\mathbf{V} \cdot \nabla c$, the vector product of the fluid velocity and the gradient of the tracer. In general, both of these vectors will be unknowns, and the nonlinearity of advection leads to many of the challenging phenomena of fluid dynamics. There are some bounds upon what advection alone can do. For variables that can be written in conservation form, e.g., mass and momentum, advection can not be a net source or sink when integrated over a closed domain.

The Lagrangian solution for a tracer comes in two parts, the parcel trajectory and the tracer value of the parcel. The tracer balance of a parcel will include the effects of a local source and of the change in volume, but crucially, the motion of a parcel does not in and of itself cause a change in tracer amplitude. An Eulerian solution comes in one piece that includes the effects of advection, $\mathbf{V} \cdot \nabla(\ )$, and so is likely to be more complex than is the corresponding, two-piece Lagrangian solution.

The method of characteristics seeks to develop an Eulerian solution of first order partial differential equations (PDEs), such as the advection equation, by defining an equivalent system of (usually simpler) ordinary differential equations (ODEs). The first step is to solve for the characteristics of the PDE, which are the parcel trajectories if the PDE is an advection equation. The second step applies the initial data to the characteristics, and the third step solves for the effects of a source acting along the characteristics. If the source vanishes, then the initial data are transported unchanged (adiabatically) along the characteristics.

More on Fig. (1). The subsurface SOFAR floats that made these trajectories were tracked acoustically. They transmit low frequency (250 Hz) pulses that are partially trapped within a sound speed minimum at about 1 km depth below the sea surface called the SOFAR channel. Trapped, low frequency sound can propagate thousands of kilometers without intersecting the surface or the sea floor. Sound pulse arrivals observed at two or more locations enable float tracking over ocean basin scales. These floats endure for up to several years, providing a long term and large space scale Lagrangian view of ocean currents. The potential vorticity balance of the wave-like motion seen here will be discussed in Part 2. These data are a small part of the subsurface float archive available online from https://www.aoml.noaa.gov/phod/ , and then search for 'float trajectories'. A Matlab script that will read and plot these data: https://www2.whoi.edu/staff/jprice/wp-content/uploads/sites/199/2023/07/Subsurffloats.txt

Author's Preface

Topics in Fluid Dynamics:

i) Lagrangian and Eulerian representations of fluid flow
ii) Dimensional analysis of models and data sets
iii) a Coriolis tutorial

This collection of three essays grew out of my experience teaching the graduate-level introduction to fluid mechanics in the MIT/WHOI Joint Program in Oceanography. Students enter the Joint Program with widely varying experience in mathematics and physics and the goal of this introductory course is to help each student learn the fundamental concepts and tools of fluid mechanics that will be an essential foundation for their research in oceanic and atmospheric sciences.

There are a number of modern, comprehensive textbooks that can serve most of the objectives of this kind of course very well. However, it seemed to me that there were three specific topics that could benefit from a greater depth of treatment than a comprehensive text can afford; these were (i) Lagrangian and Eulerian representations of fluid flow, (ii) Dimensional analysis, and (iii) the Coriolis force. The first topic is either omitted or barely mentioned in introductory texts, and a cursory treatment of the second and third topics leaves many students in a state of mystery. Of course, what is clear and sufficient for one student (or instructor) may be inadequate or unsatisfying for another student (or instructor) having a different background or level of interest. There is no doubt that the Coriolis force is described rigorously in many classical mechanics texts. However, for most physics and engineering students, the consequences of the Coriolis force are not compelling: slightly deflected rockets and very slowly precessing pendulums. But for a student of the atmosphere or ocean, the Coriolis force is vital — it shapes the structure of most large scale, low frequency flows, and so its consequences will be

encountered nearly every working day. A few additional hours devoted to a thorough-going understanding of the origin and properties of the Coriolis force will be, for them, time well spent.

With that as the backdrop, I set out to write three essays dealing with each topic in turn and with the goal of making a clear and accessible written source combined with numerical problems and software, where possible. Compared with the necessarily brief treatment of these topics found in most introductory texts, these essays go into somewhat greater depth, and at least occasionally make explicit mention of the relevant contrast class, i.e., a useful explanation must show what *is*, and it can be very helpful to consider at least a part of what *is not*, and to understand what alternatives may exist.

The second and third of these essays, 'Dimensional analysis' and 'a Coriolis tutorial', have a clearly defined scope. Examples treated in these essays are often geophysical or fluid mechanical, but the scope of dimensional analysis goes far beyond geophysics. The goal of this first essay, 'Lagrangian and Eulerian representations', is in some respects more ambitious. The overt goal is to introduce the kinematics of fluid flow by studying the transformation of observations and conservation laws between Lagrangian and Eulerian systems. An implicit, deeper goal is to address a question that lurks in the minds of many first-year students: what is it that makes fluid mechanics different from the rest of classical mechanics, and, while we are at it, why is fluid mechanics so challenging?

 I will be grateful to hear your comments and questions, and especially for suggestions that will help make these essays more accessible or more useful for your purpose.

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Contents

1 The kinematics of fluid flow.

This essay introduces some of the key concepts and mathematical tools that make up the foundation of fluid mechanics, a vast subject that encompasses widely diverse materials and phenomena. The emphasis here is upon the flow of non-exotic fluids — air and water — that make up the Earth's fluid environment.[1,2] The intermediate spatial scales and comparatively slow speeds of the atmosphere and

[1] Footnotes provide references, extensions or qualifications of material discussed in the main text. Appendices are used to include material that is relevant but probably not essential for most readers. Footnotes and appendices may be skipped on first reading.

[2] An invaluable web source for fluid dynamics generally is the collection of films by Ascher Shapiro and colleagues and specifically, 'Eulerian Lagrangian Description': https://video.odl.mit.edu/videos/591abfae9b1d432f9f9cdd05e40261da/

ocean are well within the realm of classical mechanics: conservation of mass, conservation of momentum and angular momentum and conservation of energy. The theme of this essay follows from the question — How can these conservation laws be codified in a way that is suitable for the analysis of a fluid flow?

In principle the answer is straightforward; erect a coordinate system that is suitable for describing a fluid flow, and then derive the mathematical form of the conservation laws that correspond to that coordinate system. The definition of a coordinate system is a matter of choice, and the issues are mainly in the realm of kinematics, i.e., the description of motion rather than physical properties.

1.1 What is the issue? Complex kinematics, mainly

Some day soon you will have the opportunity to study the fluid dynamics of a domain that is grand in scale and vitally important to humankind, the circulation of an ocean basin or the Earth's atmosphere. But for now you can make useful qualitative observations in a domain that is small and accessible: even the flow in a teacup will reveal some of the essential kinematic aspects of fluid flow that are common to many fluid phenomena. To initiate flow in a tea cup you need only apply a gentle, linear push on the fluid with a spoon, say, and then observe the result. The motion of the fluid bears no obvious resemblance to this simple forcing. The fluid that is pushed directly by the spoon can not simply plow straight ahead, both because water is effectively incompressible for such gentle motion and because the inertia of the fluid that would have to be displaced is appreciable. Instead, the fluid piles up in front of the spoon and then flows around both edges from front to back. The result is a pair of swirling, coherent features called vortices, or eddies, that spin in opposite directions (Fig. 2).[3] These vortices are clearly two-dimensional, despite that the forcing was a one-dimensional push. This vortex pair moves slowly through the fluid, consistent with the notion that the vortices are mutually advected by one another. Careful observation will reveal that most of the linear (one-dimensional) momentum imparted by the push is contained within their collective, translational motion. Momentum is certainly conserved, but the qualitative structure of the resulting fluid flow would be very hard to anticipate from intuition derived from solid mechanics. Still greater complexity will result if the initial push is made more vigorous; the fluid motion will spontaneously become three-dimensional and irregular, or turbulent. Thus the qualitative structure of the flow depends not only upon the geometry of the forcing, which is expected, but also upon its amplitude.

After a short time, a few tens of seconds, the smallest spatial scales of the motion will be damped by viscosity leaving larger and larger scales of motion. This damping process is in the realm of physics since it depends upon a physical property of the fluid, the viscosity, and also upon the physical scale (i.e., the size) of the flow features. Thus even though our intent in this essay is to emphasize kinematics, we can not go far with observation of real fluids before acknowledging physical phenomena. The last surviving flow feature in a tea cup forced by an impulse is likely to be a slowly revolving vortex that fills the entire domain.

What do these observations portend for a theory or description of a fluid flow? For one, every parcel[4]

[3] This and similar figures were made by the author using AI and DALL.E 2.

[4] Fluid material is variously referred to as a particle, a parcel or a patch. A fluid particle is equivalent to a solid particle in that it denotes a specific piece of the material that has a vanishing extent. If the interest is position only, then a fluid particle would suffice. A fluid parcel is a particle with a small but finite area and volume and hence can be pushed around by normal and

Figure 2: To make and observe a fluid flow you need only:

Push gently then watch,
eddies swirling left and right,
move ahead as one.

that participates in fluid flow is literally pushed and pulled by all of the surrounding fluid parcels via shear stress and normal stress (reviewed in Sec. 6.1). Thus the motion of a given fluid parcel can not be predicted in isolation from its surroundings, rather we have to predict the motion of all of the surrounding fluid parcels as well. How extensive are these so-called surroundings? It depends upon how far backward or forward in time the analysis must extend, and also upon how rapidly signals including waves are propagated within the fluid. If a parcel is followed long enough, then every parcel will have a dependence upon the entire domain occupied by the fluid. In other words, even if the goal was limited to calculating the motion of just one parcel or the flow at just one site, it will often be necessary to solve for the fluid motion over the entire domain at all times of interest. As remarked already and as you have observed (if you have studied your teacup) fluid flows will often develop motion on all accessible spatial scales, from the scale of the domain down to a scale set by viscous or diffusive properties of the fluid, typically a few millimeters in water. What was intended to be a small and simple (but unconstrained) fluid flow will thus spontaneously transform into an irregular, three-dimensional phenomenon that fills the entire, available domain and that has spatial scales much smaller and much larger than that imposed by the forcing. The tea cup and its fluid flow are well within the domain of classical physics and so everything observed is consistent with the familiar conservation laws for mass, momentum, angular momentum and energy.

It is this complex kinematics of fluid flow that most distinguishes fluid flows from the motion of otherwise comparable solid materials. The physical origin of this complex kinematics is the ease with which fluids undergo shear deformation combined with the generally small effect of viscosity on medium to large scale motions. The practical consequence is that an appropriate description and theory of fluid flow must be able to define motion and acceleration on arbitrarily small spatial scales, and that the coordinates of a fluid theory or model must, in principle, vary continuously. (Though in practice, the coordinates of numerical models are defined as finite grids, having a grid interval that then becomes an important aspect of the solution.)

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shear stresses. Parcels are the fare of the dynamical equations. A tracer patch is a macro-scale parcel that is big enough to become significantly deformed in time. The spot of cream that you may have used as the tracer in your teacup experiments is such a patch. When ‘point’ is used as a noun it will always mean a point in space, i.e., a position, rather than a fluid particle or parcel.

1.2 Two ways to observe and describe a fluid flow

Let's suppose that your task is to make a quantitative description of a fluid flow, first by observation, and then by constructing a theory or model. There are two ways to accomplish the observations, either by tracking specific, identifiable fluid material volumes that are carried about with the flow, the Lagrangian method, or by observing the fluid velocity at locations that are fixed in space, the Eulerian method (Fig. 3). Both methods are used widely in the analysis of the atmosphere and oceans, and in fluid mechanics generally. This essay aims to investigate both systems, and emphasizes the transformation of data and conservation laws from one system to the other.

1.2.1 Lagrangian coordinates are the positions and the properties of fluid parcels

The most natural way to observe a fluid flow is to observe the trajectories of small, recognizable parcels of a passive tracer that are carried along by the flow (almost certainly your observation method in the tea cup experiment). For the purpose of a continuum theory, it is essential to identify specific parcels. One possibility is to use the position of the parcels at some specified time, say the initial time, $t = 0$. Denote the initial position by $\mathbf{A}$, with Cartesian components, $(a,\ b,\ c)$. Let $\tilde{X}$ denote the position vector of a parcel whose Cartesian components are the lowercase $(\tilde{x},\ \tilde{y},\ \tilde{z})$, i.e, $\tilde{x}$ is the x-coordinate of a parcel, $\tilde{y}$ is the y-coordinate and so on. Notice that a tilde $\tilde{(\ )}$ is appended to the symbols that represent Lagrangian position to distinguish from the Eulerian position (just plain $\mathbf{X}$) coming soon. (There is no tilde with the initial position $\mathbf{A}$ since it will be reserved for the Lagrangian system.) It will be assumed that the position of parcels or floats at all later times can be observed to form a parcel trajectory,

$$\tilde{\mathbf{X}} \ = \ \tilde{\mathbf{X}}(\mathbf{A},t) \tag{1}$$

The trajectory $\tilde{\mathbf{X}}$ of specific fluid parcels is a dependent variable in a Lagrangian description (along with pressure and density) and the initial position $\mathbf{A}$ and time, t, are the independent variables. The velocity of a parcel is the time rate change of the parcel position (1) holding $\mathbf{A}$ fixed, and called the

$$\text{Lagrangian velocity:} \qquad \boxed{\tilde{\mathbf{V}}(\mathbf{A},t) \ = \ \frac{d\tilde{\mathbf{X}}(\mathbf{A},t)}{dt}\,|_{\mathbf{A}\,=\,constant} \ = \ \frac{\partial\tilde{\mathbf{X}}(\mathbf{A},t)}{\partial t}} \tag{2}$$

If instead of a parcel trajectory from a theory we have an observed float trajectory (a float is any artificial, trackable surrogate for a fluid parcel) then holding $\mathbf{A}$ constant would be replaced by holding the float identity fixed.

Eq.(2) is the first instance of the important idea of a time-derivative made while holding the parcel (or float) identity fixed, denoted by a capital D and called the

$$\text{material derivative generally:} \qquad \boxed{\frac{D(\ )}{Dt} \ \equiv \ \frac{d(\ )}{dt}\,|_{\mathbf{A}\,=\,constant}} \tag{3}$$

A material derivative D/Dt applied to a Lagrangian variable is equivalent to a partial derivative with respect to time, as in Eq. (2), since $\mathbf{A}$ is held constant. The equivalent material derivative in an Eulerian

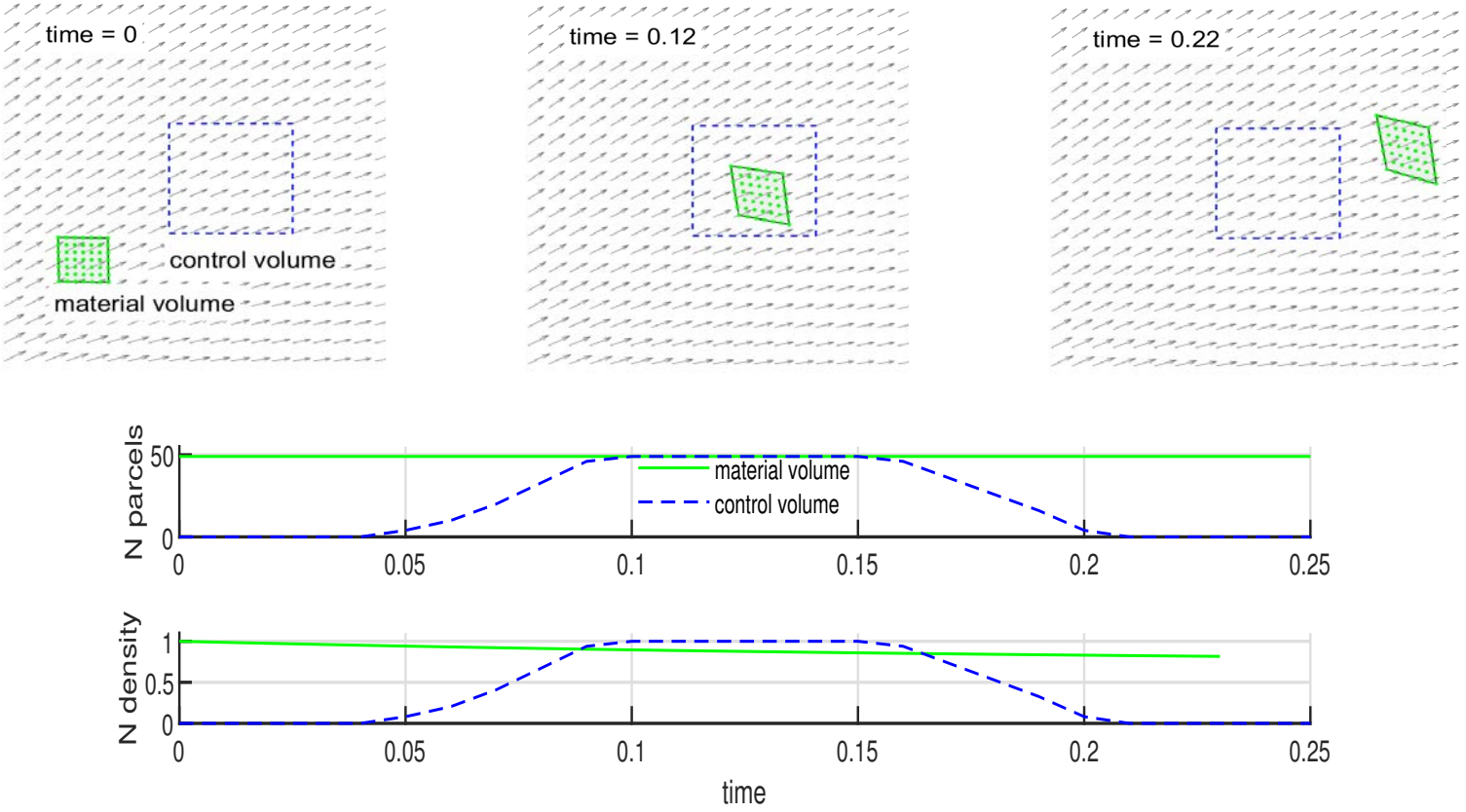

Figure 3: **(upper)** Three snapshots from a numerical tracer advection experiment that illustrate Lagrangian and Eulerian representations. A passive tracer was inserted into a velocity field given by the regular array of vectors that are directed mainly to the right and upward. This velocity field is steady, and has some divergence. It is an ideal fluid flow in that there is no diffusion. The initial condition at left includes a patch of tracer that is shaded gray with a green boundary and includes 49 passive, green parcels. This is a material volume in that it is defined by these marked fluid parcels and it moves exactly with the fluid. A control volume is defined by the dotted blue box that is fixed in space and is completely invisible to the flow. The fluid material inside the control volume is changed continually by the flow: tracer begins to enter the control volume at time = 0.05, and is flushed out by time = 0.22. **(middle panel)** Some tracer statistics. The number of parcels, N, (the 49 green dots) within the material volume (the solid green line) and within the control volume (the dashed blue line) as a function of time. The parcel count within the material volume is constant in time, since parcels do not cross the boundary of a material volume in ideal fluid flow. On the other hand, the parcel count within the control volume is strongly time-dependent due to advection of parcels through the boundaries. **(lower panel)** The parcel number density (per area) normalized by the maximum value. The parcel density within the material volume (green line) decreases with time on account of the divergence, which increases the area of the material volume, but does not change the parcel count, constant at $N = 49$. Compare this with the parcel density in the control volume (the blue dashed line) which is strongly time-dependent as parcels were advected into and then out of the control volume. A Lagrangian representation seeks to observe and predict the position, pressure and tracer properties of material volumes that move with the flow; an Eulerian representation seeks to observe and predict these fluid properties within fixed control volumes. This example illustrates that Lagrangian and Eulerian representations may be qualitatively different, even while describing the same flow.

system has a significantly different form that will be developed in Sec. 3.[5]

1.2.2 Eulerian coordinates are the fluid properties at fixed positions

If tracking fluid parcels is impractical, then the fluid velocity could be observed by means of current meters that are implanted at fixed positions, $\mathbf{X}$, the usual field coordinate (and notice there is no tilde). The essential component of every current meter is a transducer that converts fluid motion into a readily measured signal, e.g., the rotary motion of a propeller or the Doppler shift of a sound pulse. Regardless of the mechanical details, the velocity sampled in this way, termed the 'Eulerian' velocity, $\mathbf{V}$, is intended to be the velocity (2) of the fluid parcel that is present, instantaneously, within the fixed and suitably small volume of space that is sampled by the transducer. A formal definition, here said to be the Fundamental Principle of Kinematics or FPK,

$$\text{Eulerian velocity via the FPK:} \qquad \boxed{\mathbf{V}(\mathbf{X},t)\,|_{\mathbf{X}=\tilde{\mathbf{X}}(\mathbf{A},t)} \;=\; \frac{\partial \tilde{\mathbf{X}}(\mathbf{A},t)}{\partial t}} \tag{4}$$

This equation contains both Eulerian and Lagrangian variables, $\mathbf{X}$ and $\tilde{\mathbf{X}}$, and the entire content of (4) hinges upon the distinction between them. The plain (no tilde) $\mathbf{X}$ is a fixed, field coordinate. With a tilde appended, $\tilde{\mathbf{X}}$ is the position of a specific fluid parcel, here identified by the starting position, $\mathbf{A}$. No doubt the FPK Eq. (4) looks a bit abstract, but it codifies a clear and concrete meaning: the (Eulerian) fluid velocity at a given time and at a fixed position $\mathbf{X}$ is the (Lagrangian) velocity of the fluid parcel that happens to be present at that position, at that instant in time. The velocity $\mathbf{V}$ is a dependent variable in an Eulerian description, as are pressure and density (and possibly other thermodynamic properties) and the independent variables are the position, $\mathbf{X}$, and time, t. The Eulerian velocity defined by the FPK may be seen as a composite function constructed from the Lagrangian velocity and position (composite functions are reviewed in an appendix, Sec. 6.2).[6]

One way to appreciate the difference between the Lagrangian velocity $\tilde{\mathbf{V}}$ and the Eulerian velocity $\mathbf{V}$ is to note that $\tilde{\mathbf{X}}$ in the Lagrangian velocity of Eq. (2) is the position of a moving parcel, while $\mathbf{X}$ in Eq. (4) is the arbitrary and fixed position of a current meter. Parcel position is a result of the fluid flow rather than our choice, aside from the initial position. As time runs, the position of any specific parcel will change, barring that the flow is static. The velocity observed at the current meter position will be the velocity of the sequence of parcels (each having a different $\mathbf{A}$) that move through that position as time runs. In general, the Lagrangian and Eulerian velocities are different, and in many cases they will be qualitatively different, as illustrated by the tracer advection experiment of Fig. (3).[7] However, there are specific, related times and places — given by the FPK — where the Lagrangian and Eulerian velocities and other fluid properties are equal.

[5] There are about twenty boxed equations in this essay, beginning with Eq. (2), that you are likely to encounter over and over again in a study of fluid mechanics.

[6] The present usage FPK is idiosyncratic, and you might never see it elsewhere. The claim that the FPK is 'fundamental' is sustainable if, as presumed here in Part 1, the starting point of an analysis is a (Lagrangian) trajectory.

[7] In some fields it is common to refer to transport by fluid motion as 'convection'. Convection is also used to refer to fluid motion caused specifically by density differences. In the end, the terms 'advection' and 'convection' are synonymous.

The float and current meter data of Fig. (1) afford an opportunity to check the FPK in practice: when the flow is smoothly varying on the horizontal scale of the float cluster, and when the floats surround the current meter mooring, the Lagrangian velocity (the blue comets are five-day long trajectories) and the Eulerian velocity (the single red vector) appear to be very similar. But at other times, and especially when the velocity is changing direction rapidly in time or in space, the equality expected from the FPK is not obvious.[8]

The present use of Lagrangian and Eulerian is standard. If no such label is appended, then Eulerian is almost always understood to be the default.[9] This Lagrangian/Eulerian usage should not be construed to mean that there are two physical fluid velocities. Rather, a given fluid flow can be sampled in two quite different ways, by tracking specific parcels or foats (Lagrangian), or by observing the velocity of the sequence of fluid parcels that flow through a fixed site (Eulerian). The formal statement of the relationship between these systems is the FPK, Eq. (4). Much of what follows in this essay amounts to variants or extensions of the FPK combined with the familiar conservation laws of classical physics.

1.3 The goal and the plan — Lagrangian to Eulerian

Now that we have learned (or imagined) how to observe a fluid flow at one or a few parcels or sites, we can begin to think about surveying the entire domain. This will require an important decision regarding the sampling method; should these observations come from tracking a large number of fluid parcels (or floats) as they wander throughout the domain, or, by deployment of current meters at many additional sites? In principle, either approach could suffice to define the flow if done in exhaustive detail (an example being the solution from the ocean circulation model of Fig. 5). Nevertheless, the observations themselves and the analysis needed to understand these observations would be quite different, as will be discussed in examples below.

In practice, the choice of a sampling method will probably be decided more by the availability of floats or current meters, then by any Lagrangian or Eulerian preference that you may happen to hold. Thus it commonly happens that observations are made in one system, and then diagnostic analysis is performed in the other system. A similar kind of duality arises in the development of models and theories; the (Lagrangian) parcels of a fluid flow follow conservation laws that are identical with those followed by the solid particles of classical dynamics. Nevertheless, the theory applied to a continuum model of fluid flow is almost always Eulerian. The goal of this essay is to help you develop some

[8] If a model seems to be consistent with relevant observations, then there may not be much more to say. Potentially much more interesting would be an outright failure of a plausible theory. What would we do here if the float and current meter velocities did *not* appear to be similar? We would not lay the blame on Eq. (4), which is, in effect, an identity, i.e., it defines what we mean by the Eulerian velocity. Instead, we would start to question, in roughly this order: 1) If $\tilde{\mathbf{X}} = \mathbf{X}$ as required by the FPK, since this would imply a collision between float and current meter (none was reported). 2) If some time-averaging had been applied. It was, inevitably, and time-averaging can have a surprising effect that will be taken up in Part 2. 3) Whether the float tracking accuracy was sufficient, and whether the current meter had been improperly calibrated or had malfunctioned.

[9] This usage is not the least bit descriptive of the two systems in the way that 'material' and 'field' are, somewhat. It is also inaccurate as historical attribution; Lamb, *Hydrodynamics*, 6th ed., (Cambridge Univ. Press, 1937) credits Leonard Euler with developing both representations. Whether good history or not, this essay propagates the Lagrangian and Eulerian usage because it is so thoroughly ingrained into the topic that any attempt to make a change would cause more confusion than benefit.

understanding of both systems, and to appreciate how Lagrangian and Eulerian data, models and concepts may be woven together to implement the analysis of fluid flow.

The point of view and the plan taken by this essay is that Lagrangian observation and conservation properties are the starting point, Sec. 2. However, this felicity does not extend all the way to the three-dimensional Lagrangian equations of motion, which are awkward for most applications. The object of Sec. 3 is then to derive the Eulerian equations of motion starting from a Lagrangian perspective. The powerful method of characteristics, taken up in Sec. 4, uses the properties of advection to construct a system of ordinary differential equations that are the equivalent of a first order, partial differential equation. This can lead to an insightful analysis of advection phenomenon, e.g., shock waves. Sec. 5 is a very brief preview of the topics considered in Part 2. Sec. 6 is made up of two appendices. Sec. 6.1 reviews the physical properties of fluids, and the response to normal and tangential stress. Sec. 6.2 reviews the mathematical properties of composite functions. Each section is concluded with a list of key ideas and a few homework problems or projects. An overall summary is provided by the Abstract.

1.4 The key ideas of this section

1) The distinctive property of a fluid is that it will *flow* when subjected to a shear stress or an unconstrained normal stress (reviewed in the appendix of Sec. 6.1).

2) Even when the physics is purely classical, the kinematics of even a simple (but unconstrained) fluid flow can be daunting, and requires coordinate systems that are adequate to a continuous medium.

3) The motion of fluid parcels is entangled in the sense that the motion of any one fluid parcel is determined by the pressure and shear stress exerted by all of the neighboring parcels. Given sufficient time, the relevent neighborhood will extend all the way to the domain boundary.

4) A given fluid flow may be described and modelled in one of two ways. A Lagrangian description follows the motion of specific, moving fluid volumes called parcels. An Eulerian description is of velocity and other fluid properties at fixed points in space.

5) The Eulerian velocity at a given $(\mathbf{X}, t)$ is the (Lagrangian) velocity of the parcel moving through that $\mathbf{X}$ and at that time, i.e., instantaneously in space and time. The formal relation between Eulerian and Lagrangian velocity is here said to be the Fundamental Principal of Kinematics, or FPK, and which amounts to a composite function constructed from Lagrangian trajectories.

6) A fluid flow may be defined fully by a sufficient number of either Lagrangian or Eulerian measurements. In many investigations it will be helpful or necessary to transform data from one system to the other, e.g., Lagrangian observations may be mapped into an Eulerian field via the FPK. The reverse transformation, Eulerian velocity field to Lagrangian trajectories, is an essential diagnostic of material (tracer) transport and requires an integration (taken up in Part 2).

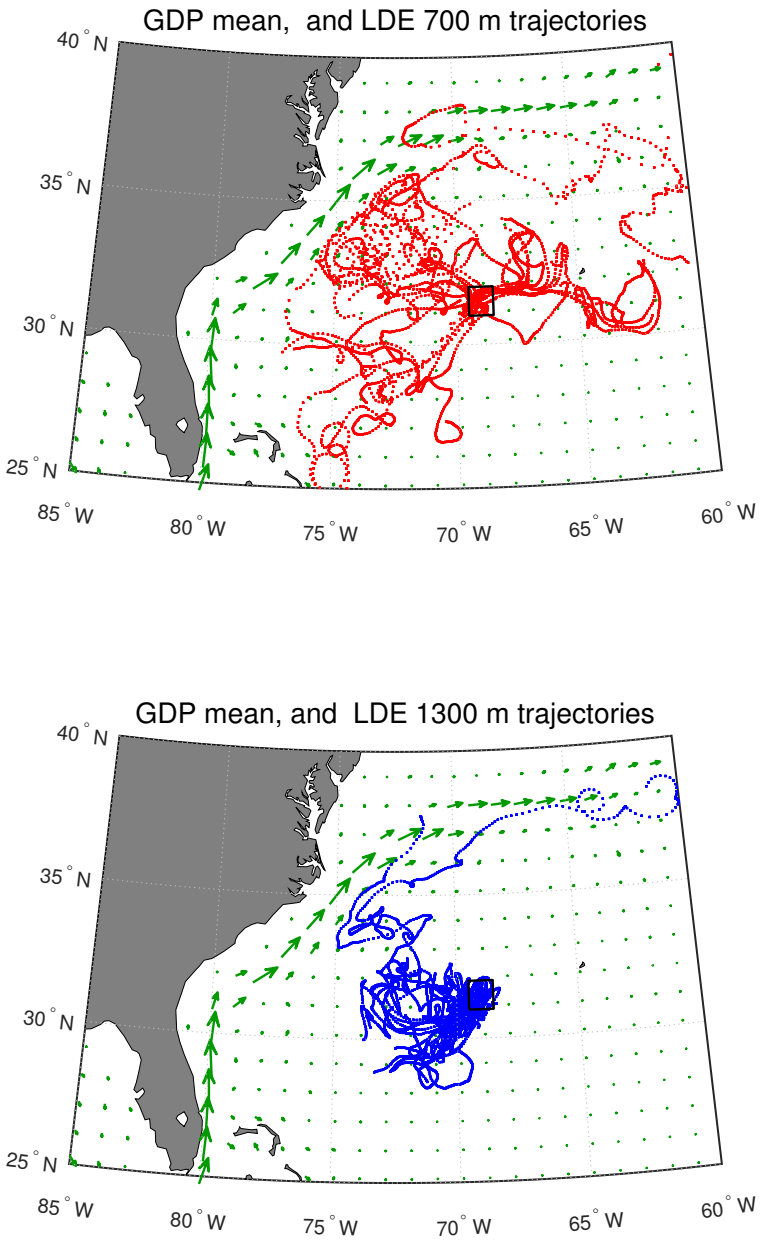

Figure 4: Trajectories of acoustically-tracked SOFAR floats from a depth of 700 m **(upper)** and 1300 m **(lower)** that were deployed as part of the Local Dynamics Experiment (LDE). (A subset of this data is in Fig. 1.) Floats were launched in the small rectangular region near 32 N, 69 W, which is the central Sargasso Sea. The dots are at one day intervals, and the trajectories are of variable duration, nominally one year. The regularly-spaced array of green vectors is the time-mean near-surface velocity estimated over roughly three decades by the Global Drifter Program (GDP, Sec. 2.4). There is some, limited qualitative similarity between these two very different data sets, but also striking differences. Both reveal a comparatively swift, generally northeastward flow along the northern and western edge of the Sargasso Sea, the Gulf Stream. In the GDP long-term mean, the Gulf Stream looks to be laminar. However, the instantaneous Gulf Stream that is evident in several of these SOFAR float tracks meanders vigorously, sometimes forming complete loops called Gulf Stream rings. Within the central Sargasso Sea, the time-mean flow is westward and very small compared to the time-dependent currents associated with mesoscale eddy variability, which has a time scale (period) of several months. The complex and entangled trajectories of the mid-ocean flow have been characterized as spaghetti diagrams (Prof. Tom Rossby, personal communication).

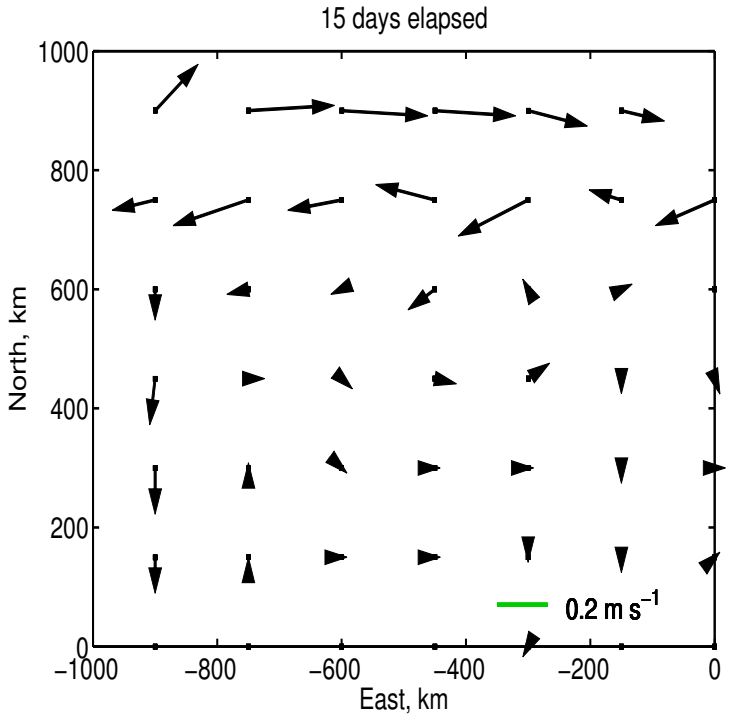

Figure 5: An ocean circulation model solved by an Eulerian system, and then sampled for the Eulerian velocity (the regularly spaced black vectors) and for a comparable number of parcel trajectories (green comets that will appear in the animation linked below). The domain is a square basin 2000 km by 2000 km driven by a basin-scale wind having negative curl, as if a subtropical gyre. Only the northwestern quadrant of the model domain and only the upper most layer of the model are shown here. The main circulation feature is a rather thin western and northern boundary current that flows clockwise. There is also a well-developed westward recirculation just to the south of the northern boundary current. This westward flow is (baroclinically) unstable and oscillates with a period of about 60 days, comparable to the period of the north-south oscillation of the float cluster seen in the cover graphic. This particular model solution suffers from poor horizontal resolution, the grid interval being one fourth the interval between velocity vectors plotted here. As one consequence, the simulated fluid must be assigned an unrealistically large viscosity, more like cold molasses than water (viscosity is reviewed in Sec. 6.1). A result is less variance in small scale features than is realistic for the ocean, but as much as this model grid can resolve. This ocean model is available from the author's web page. To see these data animated, visit https://www2.whoi.edu/staff/jprice/wp-content/uploads/sites/199/2022/06/basinmovie.mp4

1.5 Homework problems and projects

1) How many observation points do you estimate would be required to define completely the fluid flow in a teacup? In particular, what is the smallest spatial scale on which there is a significant variation of the fluid velocity? Does the number depend upon the state of the flow, i.e., whether it is weakly or strongly stirred? Does it depend upon time in any way? The viscosity of water varies by a factor of about four as the temperature varies from 0 to 100 Centigrade. Can you infer the sense of this viscosity variation from your observations? To achieve a much larger range of viscosity, consider a mixture of water and honey. What fundamental physical principles, e.g., conservation of momentum, second law of thermodynamics, can you infer from purely qualitative observations?

2) The fluid motion in a teacup may also include waves: capillary waves have short wavelengths, only a few centimeters, while gravity waves can have any larger wavelength, and may appear mainly as a sloshing back and forth over the entire tea cup. Waves can propagate momentum and energy much more rapidly than can the vortices of Fig. 2. Capillary and gravity waves owe their entire existence to the free surface, and may not appear at all if the speed at which the spoon is pushed through the fluid does not exceed a certain threshold. Can you estimate roughly what that speed is? It may be helpful to investigate this within a somewhat larger container.

3) To verify that air and water *are* Newtonian (stress proportional to deformation rate; viscosity and stress are reviewed in Sec. 6.1) requires rather precise laboratory measurements that are not likely to be accessible. But, it is very helpful to understand what a Newtonian fluid *is not,* and there are many non-Newtonian fluids that are encountered routinely. High molecular weight polymers such as paint and mayonnaise are said to be 'shear-thinning'. Under a small stress these materials may behave like very weak solids, i.e., they will deform but not quite flow until subjected to a shear stress that exceeds some threshold that is often an important characteristic of the material. 'Shear-thickening' fluids are less common, and can appear to be quite bizarre. Here's one you can make at home: a solution of about three parts cold water and two parts of corn starch powder will make a fluid that flows under a gentle stress like a very viscous fluid. But when the corn starch solution is pushed too vigorously, it will suddenly seize up, forming what seems to be a (weak) solid. Try adding a drop of food coloring to the cold water, and observe how or whether the dyed material can be stirred and mixed into the remainder. Sketch the qualitative stress/deformation (or rate of deformation) relationship for these non-Newtonian fluids, as in Figs. (21) and (22). From the vantage of a small water bug: does the water appear to be more or less viscous than it does to a human-sized swimmer?

1.6 About this essay

Readers are presumed to be familiar with basic concepts of classical mechanics and with multivariable calculus. Some prior exposure to fluid mechanics will be very helpful as well, but less for the technical details than for the all-important motivation — if a passing mention of Lagrangian observations or Lagrangian models has piqued your interest, then this essay might be for you.

The intent for this essay is educational. This essay is not a research monograph, nor is it a research review, but there are links to selected research articles. It is best used as a supplement (rather than as a substitute) to a comprehensive fluid mechanics text. The primary reason is that the plan (outlined in Sec. 1.3) is to start with a Lagrangian perspective, and then transform step by step to the Eulerian system most

often used for theory. The shortest and easiest path to useful results is instead an Eulerian treatment from the outset, and which is the plan of most introductory textbooks.[10]

Financial support during the first preparation of this essay in 2006 was provided by the Academic Programs Office of the Woods Hole Oceanographic Institution. Additional salary support was provided by the U.S. Office of Naval Research. Administrative support during the revision of 2023 was provided by Woods Hole Oceanographic Institution.

This essay is dedicated to Prof. Peter Niiler, 1937 - 2010, Scripps Institution of Oceanography, whose enthusiasm, creativity and dedication to ocean science made possible the Global Drifter Program discussed in Sec. 2.4.1.

2 The Lagrangian (or material) coordinate system.

One helpful way to envision a fluid flow is as an ongoing coordinate transformation that maps parcels from a starting position, $\mathbf{A}$, into the positions $\tilde{\mathbf{X}}$ at later times. A sequence of positions $\tilde{\mathbf{X}}(\mathbf{A}, t_1)$, $\tilde{\mathbf{X}}(\mathbf{A}, t_2)$ ... $\tilde{\mathbf{X}}(\mathbf{A}, t_n)$ is a trajectory, Eq. (1). Each trajectory must be tagged with a unique $\mathbf{A}$, which is thus a constant on that trajectory, and serves to identify the parcel. It is assumed that the mapping from $\mathbf{A}$ to $\tilde{\mathbf{X}}$ is continuous in that adjacent parcels will never be split apart, and neither will one parcel occupy the same position as another parcel. The formal statement of this is that the determinant of the Jacobian of the transformation $\mathbf{A}$ to $\tilde{\mathbf{X}}$ does not vanish.[11] A physical interpretation is that the fluid density does not vanish or become infinite (Sec. 2.4). With these conventional assumptions in place, the mapping of parcels from initial to subsequent positions, Eq. (1), can be inverted so that a Lagrangian representation

[10] Excellent, modern examples include P. K. Kundu and I. C. Cohen, *Fluid Mechanics* (Academic Press, 2001), by B. R. Munson, D. F. Young, and T. H. Okiishi, *Fundamentals of Fluid Mechanics, 3rd ed.* (John Wiley and Sons, NY, 1998), by D. C. Wilcox, *Basic Fluid Mechanics* (DCW Industries, La Canada, CA, 2000) and by D. J. Acheson, *Elementary Fluid Dynamics* (Clarendon Press, Oxford, 1990). A superb text that emphasizes experiment and fluid phenomenon is by D. J. Tritton, *Physical Fluid Dynamics* (Oxford Science Pub., 1988). Two other classic references, comparable to Lamb but more modern are by Landau, L. D. and E. M. Lifshitz, 'Fluid Mechanics', (Pergamon Press, 1959) and G. K. Batchelor, 'An Introduction to Fluid Dynamics', (Cambridge U. Press, 1967). An especially good discussion of the physical properties of fluids is Ch. 1 of Batchelor's classic text. A rather advanced source for fluid kinematics is Chapter 4 of Aris, R., *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, (Dover Pub., New York, 1962). A superb monograph that goes far beyond the present discussion of Lagrangian models is by A. Bennett, *Lagrangian Fluid Dynamics*, Cambridge Univ. Press, 2006.

[11] The important Jacobian operator arises on several occasions. An excellent introduction to and interpretation of the Jacobian is available online thanks to the Khan Academy, https://youtu.be/p46QWyHQE6M

can be transformed to an Eulerian representation,

$$\underbrace{\tilde{\mathbf{X}} = \tilde{\mathbf{X}}(\mathbf{A},t)}_{\text{Lagrangian representation}} \quad \Longleftrightarrow \quad \underbrace{\mathbf{A} = \mathbf{A}(\tilde{\mathbf{X}},t)}_{\text{(part way to) Eulerian representation}} \tag{5}$$

at least in principle. In a Lagrangian representation, the starting position, $\mathbf{A}$, is the independent spatial variable, and is presumed known, and the subsequent positions $\tilde{\mathbf{X}}$ is the dependent variable.

The essence of (5) is that a given $\mathbf{A}$ and time map to a unique $\tilde{\mathbf{X}}$. When that holds, then the mapping can be reversed by changing the sign of the velocity. Transport by a fluid velocity is deterministic and time-reversible. A fluid model that considers advection alone is an idealization that is often and aptly called an 'ideal fluid', or sometimes an Euler fluid. A more realistic fluid model will acknowledge also diffusion, either by molecular processes (the viscosity of Sec. 6.1), or by fast time scale, unresolved motions (discussed in Part 2, Sec. 4). In the presence of diffusion, a given parcel does not map into a unique position, but rather into a cloud whose radius grows as the square root of the elapsed time. Diffusion is stochastic and is not time-reversible.; diffusion is thus not consistent with Eq. (5). This essay emphasizes fluid flow, and omits diffusion unless specifically noted.

2.1 A simple Lagrangian flow

All this talk of coordinate transformations is rather abstract — it is time for an example of an ideal Lagrangian flow. For the present purpose it is appropriate to consider a one-dimensional domain denoted by R^1. Compared with a three-dimensional domain, R^3, this minimizes algebra greatly and so helps to highlight the important features of a Lagrangian description. However, there are important aspects of a three-dimensional flow that are not contained in one space dimension, and so this will have to be extended to three dimensions in Sec. 2.4. But for now, assume to know the trajectories of all the parcels in a one-dimensional domain with spatial coordinate x, and non-dimensional time, $t \geq 0$, by way of the explicit formula

$$\tilde{x}(a,t) = U_o\, t + a(1+2t)^{1/2}, \tag{6}$$

where U_o is a spatially-uniform velocity and a is the initial position. Once we identify a parcel by specifying the starting position, $\tilde{x}(t=0) = a$, this handy little formula gives the position, $x = \tilde{x}$, of that specific parcel at any later time. It is most unusual to have so much information presented in such a convenient way, and in fact, this particular flow has been concocted to have just enough complexity to be interesting for the present purpose, but has no physical significance. Notice that there are no parameters in Eq. (6) that give any sense of a physical length scale or time scale, i.e., whether this is meant to describe the flow in a capillary or an ocean basin.

The (Lagrangian) velocity of a parcel is readily calculated as the time derivative of the position holding a constant,

$$\tilde{u}(a,t) = \frac{\partial \tilde{x}}{\partial t} = U_o + a(1+2t)^{-1/2}, \tag{7}$$

and the acceleration is just

$$\frac{\partial^2 \tilde{x}}{\partial t^2} \;=\; \frac{\partial \tilde{u}}{\partial t} \;=\; -a(1+2t)^{-3/2}. \tag{8}$$

These are the same thing as the velocity and the acceleration of a solid particle, which is part of the appeal of the Lagrangian system.

Given the initial positions of four parcels, say a = (0.1, 0.3, 0.5, 0.7), the trajectories and velocities follow from Eqs. (6) and (7) (Figs. 6 upper and middle). Note that the velocity depends upon the initial position, a. If $\tilde{u}$ did not depend upon a, then the flow would necessarily be spatially uniform, i.e., all the fluid parcels in the domain would have exactly the same velocity, U_o. The flow shown here has the following form: all parcels shown (and all of the fluid in $a > 0$) are moving in the direction of positive x; parcels that started at larger a move a little faster, Eq. (7). All of the parcels having $a > 0$ are also decelerating and the magnitude of the deceleration increases with a, Eq. (8). If the density remained nearly constant, which it does in most geophysical flows but does not in the one-dimensional flow defined by Eq. (6), then it would be appropriate to infer a force directed in the negative x direction (more on this below).

The Lagrangian representation Eq. (6) shows, in the most straightforward way possible, where fluid parcels released into a flow at the initial time and position $x = a$ will be found at some later time. If the need was to observe how a fluid flow carried a pollutant from a source (the initial position) into the rest of the domain, then this Lagrangian representation would be ideal; we could simply release or tag parcels over and over again at the source position, and then observe or calculate where the parcels were carried by the flow.

2.2 Mass and momentum conservation in a Lagrangian system

This section will develop the Lagrangian forms of mass and momentum conservation that are required for a forward calculation in the Lagrangian system, i.e., to predict rather than to analyze or observe a fluid flow. This will continue with a one-dimensional geometry until Sec. 2.4. As above, this assumes an ideal fluid so that all effects of diffusion are omitted, including the effects of viscosity (momentum diffusion).

In the simple flow defined by (6) the velocity is solely in the x-direction, and all variations of the pressure, fluid density, etc., are in the x-direction only (Fig. 7). Let's follow a material (linear) volume of fluid that occupies the interval $a_1 < x < a_2$ in the initial state. The cross-sectional area of this material volume is denoted by H and is presumed constant, as if a pipe flow. At some later time, this volume will be displaced to a new position where its endpoints will be at $x = \tilde{x}_1$ and $x = \tilde{x}_2$.

Mass conservation. The mass of the volume in its initial state is just

$$M_o \;=\; H\bar{\rho}_0(a)(a_2 - a_1), \tag{9}$$

where the overbar indicates mean value. After the material volume is displaced, the end points will be at $\tilde{x}_1(a_1,t)$, etc., but the mass in the displaced position will be unchanged,

$$M \;=\; H\bar{\rho}(a,t)(\tilde{x}_2 - \tilde{x}_1) \;=\; M_o. \tag{10}$$

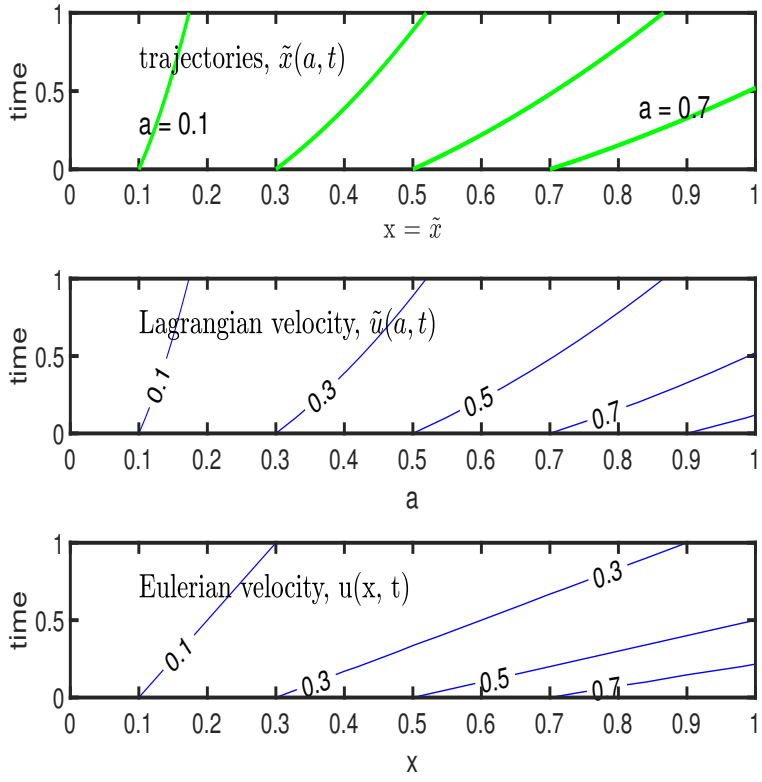

Figure 6: Lagrangian and Eulerian representations of the one-dimensional, time-dependent flow defined by the trajectories of Eq. (6). **(upper)** The solid green lines are the trajectories $\tilde{x}(a,t)$ of four parcels whose initial positions were $a = 0.1$, 0.3, 0.5 and 0.7 (left to right). **(middle)** The Lagrangian velocity, $\tilde{u}(a,t) = \partial\tilde{x}/\partial t$, as a function of initial position, a, and time. The solid blue lines are contours of constant velocity, not trajectories as above. **(lower)** The corresponding Eulerian velocity field $u(x,t)$ dicussed in Sec. 3.1, and again the lines are contours of constant velocity. These three graphs look similar because of the very simple geometry of the trajectories, Eq. (6), but they are three quite different things.

How can we be so sure? Because in this presumed ideal fluid, the parcels that make up the volume can not move through one another or through the parcels on the end points that mark the boundaries. Thus the material contained within this volume remains the same under fluid flow and hence the name 'material volume'; a two-dimensional example was sketched in Fig. (3). Equating the masses in the initial and subsequent states,

$$\bar{\rho}_0(a)(a_2 - a_1) \;=\; \bar{\rho}(a,t)(\tilde{x}_2 - \tilde{x}_1),$$

shows that the density of the parcel at later times is related to the initial density by

$$\bar{\rho}(a,t) \;=\; \bar{\rho}_0(a)\frac{a_2 - a_1}{\tilde{x}_2 - \tilde{x}_1}.$$

If the interval of Eqs. (9) and (10) is small, and assuming that ρ is smoothly varying, then the ratio of the

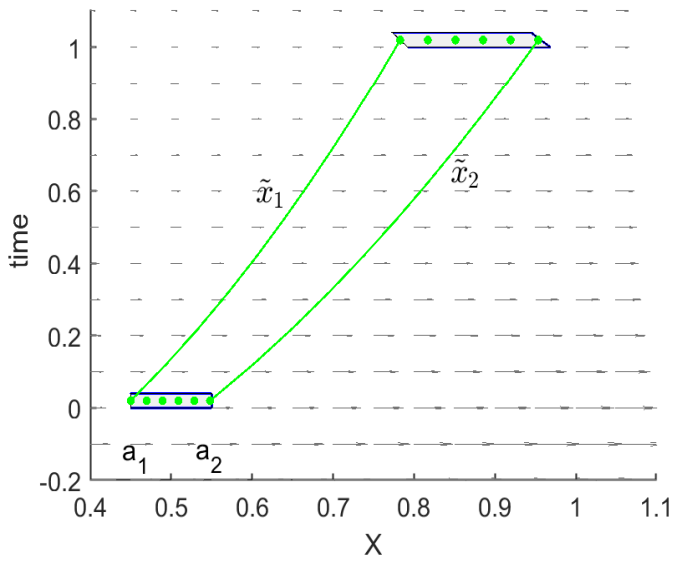

Figure 7: The one-dimensional velocity field $u(x,t)$ of Eq. (26) is plotted as the array of velocity vectors. The grey bar represents a material line whose end points are the parcels at $\tilde{x}_1$ and $\tilde{x}_2$ and whose trajectories $\tilde{x}(a,t)$ are the green lines. This velocity field is divergent and hence the length of the material line increases in time; the mass density decreases in inverse proportion. The parcels that make up this one-dimensional material volume maintain their relative order.

lengths becomes the partial $\tilde{x}$ derivative, and

$$\text{Lagrangian mass conservation in 1-d:} \quad \boxed{\tilde{\rho}(a,t) \;=\; \rho_0(a)\left(\frac{\partial \tilde{x}}{\partial a}\right)^{-1}} \tag{11}$$

which holds for an ideal fluid (no diffusion and no exchange with the exterior). This is exact, since no terms involving products of small changes have been dropped. The important term here is $\partial\tilde{x}/\partial a$, called the linear deformation. In the case sketched in Fig. (7), the displacement increases in the direction of increasing a, and hence $\partial\tilde{x}/\partial a > 1$ and this fluid flow is accompanied by an increase in the volume (here just length) of a parcel, compared with the initial state.

A convenient, normalized measure of the linear deformation is the condensation,

$$C(a,t) \;=\; \left(\frac{\partial \tilde{x}}{\partial a}\right)^{-1} - 1 \;=\; \frac{\rho-\rho_o}{\rho_o}. \tag{12}$$

If $\frac{\partial\tilde{x}}{\partial a} > 1$ so that the parcel has been stretched, then $C < 0$ and the associated fractional density change will be negative. For small amplitude motions, e.g., most acoustic waves, the condensation hovers around zero. The condensation that accompanies the simple Lagrangian flow Eq. (6) is

$$C(a,t) \;=\; \frac{1}{(1+2t)^{1/2}} - 1 \;\leq\; 0$$

and is not small for $t \;=\; O(1)$. This is an artifact of a one-dimensional flow: if a one-dimensional flow has some spatial variation then it will have a comparably large condensation; a two- or three-dimensional flow is not so constrained. As noted in Sec. 2.4, the three-dimensional version of the linear deformation is the change in volume of a unit volume, which is estimated by the determinant of the Jacobian of the transformation $\mathbf{A} \;\rightarrow\; \tilde{\mathbf{X}}$.

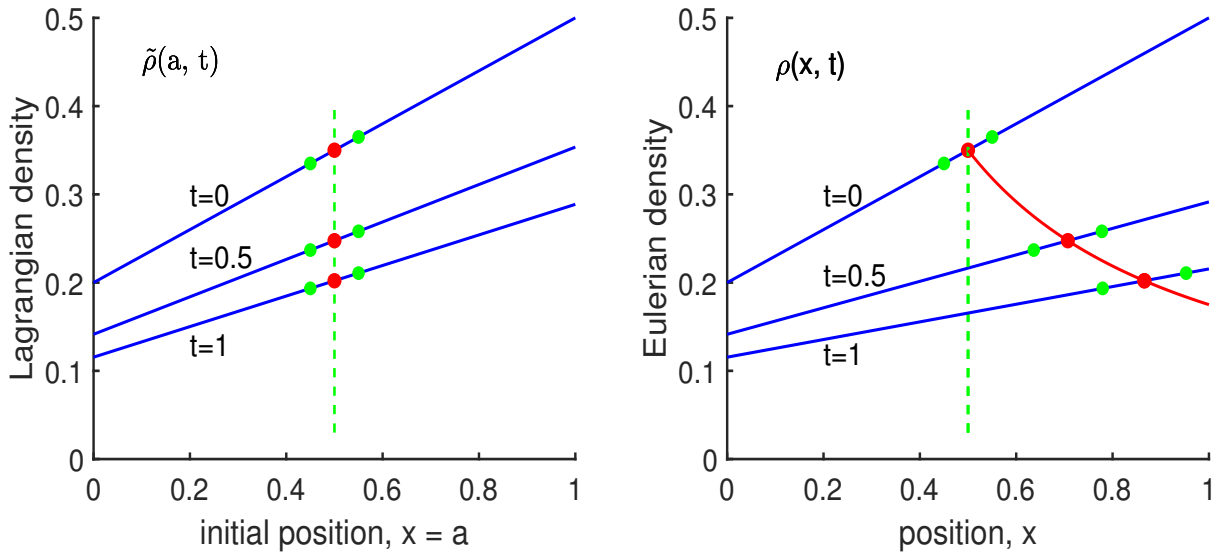

Figure 8: **(left)** A Lagrangian representation of density $\tilde{\rho}(a,t)$ given by Eq. (14) and evaluated at three times, $t = 0, 0.5, 1$. The red and green dots represent three parcels, $a = 0.45, 0.5, 0.55$. These parcels do not move in these Lagrangian coordinates (a,t), but the density decreases in time due to the linear deformation (stretching) shown in the previous figure. **(right)**. The Eulerian representation of this density (Eulerian conservation equations are discussed in detail in Sec. 3). The same three parcels are shown here too, and they move in this coordinate space; the red curve is the trajectory in (x, ρ) of the center parcel. The Eulerian density $\rho(x,t)$ decreases with time due to stretching (the distance between the three parcels increases with time), and due to advection of the density profile toward larger x (the parcels shift rightward with increasing time).

This one-dimensional Lagrangian statement of mass conservation shows that density changes of a material volume are inversely related to the linear deformation. Thus when a material volume is stretched (expanded) compared with the initial state, the case shown schematically in Fig. (7), the density of the fluid within that volume will necessarily be decreased compared with ρ_0. Indeed, in this one-dimensional ideal fluid model, the only way that the density of a material volume can change is by linear deformation. It is important to understand that motion alone — that is, a spatially-uniform motion without deformation, e.g., the U_o of Eq. (7) — will not cause a change in the density.

As an example of density represented in a Lagrangian system, assume an initial density distribution

$$\rho_0(a) \;=\; \rho_c + \Gamma a, \tag{13}$$

with ρ_c and Γ positive constants, that is embedded in the Lagrangian flow, Eq. (6), that produces trajectories $\tilde{x} \;=\; U_o\, t \;+\; a(1+2t)^{1/2}$ and linear deformation $\partial\tilde{x}/\partial a \;=\; (1+2t)^{1/2}$. By Eqs. (11) and (13) the Lagrangian density of this flow evolves as

$$\tilde{\rho}(a,t) \;=\; \frac{\rho_c + \Gamma a}{(1+2t)^{1/2}}, \quad \text{and} \quad \tilde{x}(a,t) \;=\; U_o\, t \;+\; a(1+2t)^{1/2}. \tag{14}$$

This density is evaluated for $\rho_c = 0.2$, $\Gamma = 0.3$, $U_o = 0$ and at three times in Fig. (8), left, the blue lines.

The density at constant a, i.e., the density of a given parcel, decreases with time on account of the linear deformation of this flow, as in Fig. (7).

Fig. (8) also includes three parcels identified by their initial positions and corresponding densities. In the Lagrangian coordinates, $\tilde{\rho}(a,t)$, these parcels are stationary, despite that they move in physical space, (x,t). If there was no deformation, then the density would be constant in time on each parcel: as noted above, motion alone (absent deformation) does not change the mass density of a parcel (and omitting diffusion). However, knowing only the density of a parcel but not its position, would be incomplete for most purposes. A useful Lagrangian solution will be the density $\tilde{\rho}(a,t)$, and the trajectory, $\tilde{x}(a,t)$, written alongside as in Eq. (14). Thus the Lagrangian solution comes in two parts: the tracer property on moving parcels (mass density in this case), as well as the trajectory of those parcels.

Momentum conservation. As noted in Sec. 2.1, a fluid parcel is presumed to follow Newton's laws of classical mechanics. For the present purpose, pressure is presumed to be the only force acting upon a material volume, i.e., there is no viscosity and no other external force. Pressure is important in almost every flow, but small scale flows may also be effected significantly by viscous effects. A fluid model that makes use of these idealizations (no diffusion and pressure forces only) is variously referred to as a perfect fluid, an ideal fluid, or sometimes an Euler fluid.

Consider the (linear) volume of Fig. (7). The pressure force on the right side of this volume will be $-P(\tilde{x_2})H$ and the pressure force on the left side $P(\tilde{x_1})H$ (the sign changes with the unit normal). The net pressure force on the volume is then the sum, $F = -(P(\tilde{x}_2) - P(\tilde{x}_1))H$. The acceleration is $\frac{\partial^2 \tilde{x}}{\partial t^2}$ and the mass of the material volume is just $M = \rho H(\tilde{x}_2 - \tilde{x}_1)$. Classical momentum conservation (mass $\times$ acceleration = force) applied to this material volume is then

$$\rho H(\tilde{x}_2 - \tilde{x}_1)\frac{\partial^2 \tilde{x}}{\partial t^2} \;=\; -(P(\tilde{x}_2) - P(\tilde{x}_1))H.$$

Dividing by the $\tilde{x}$ interval $\tilde{x}_2 - \tilde{x}_1$, and taking the limit procedure above gives

$$\text{Lagrangian momentum conservation in 1-d:} \quad \boxed{\frac{\partial^2 \tilde{x}}{\partial t^2} \;=\; -\frac{1}{\rho}\frac{\partial P}{\partial \tilde{x}}} \tag{15}$$

This may look familiar on first sight (just like the pressure gradient you may know from basic fluid mechanics, and that will arise in the Eulerian system, Sec. 3.4.3), but notice that this gradient of pressure is with respect to $\tilde{x}$, which is a dependent variable in a Lagrangian system. This becomes awkward in two or three space dimensions (more on this in Sec. 2.4).

2.3 An exact wave equation in one dimension

An equation of state, $P = P(\rho)$, or, $\rho = \rho(P)$ (Eq. (126) of the appendix, Sec. 6.1), provides a third and crucial piece of information by linking pressure with density, and we already know that the density is related inversely to the linear deformation. Thus the system of equations, Eqs. (126), (11) and (15) are in principle a complete set for the three unknown variables $\tilde{x}$, ρ and P. It is often useful to reduce such a system of equations to a single equation in one variable, if possible, and in this one-dimensional system it

is straightforward. The pressure term may be eliminated from the momentum equation (15) by substituting the derivative of pressure from an equation of state,

$$\frac{\partial P(\rho)}{\partial a} = \frac{dP(\rho)}{d\rho}\frac{\partial \rho}{\partial a}.$$

The derivative of density may in turn be eliminated by using the mass conservation relation, Eq. (11),

$$\frac{\partial \rho}{\partial a} = -\rho_o \frac{\partial^2 \tilde{x}}{\partial a^2}/(\frac{\partial \tilde{x}}{\partial a})^2.$$

Substitution into the momentum equation then yields an exact, nonlinear wave equation in one dimension:

$$\frac{\partial^2 \tilde{x}}{\partial t^2} = \frac{\frac{dP}{d\rho}(\rho_0 \frac{\partial \tilde{x}}{\partial a})}{(\frac{\partial \tilde{x}}{\partial a})^2}\frac{\partial^2 \tilde{x}}{\partial a^2}. \tag{16}$$

It is nonlinear because the term $(\frac{\partial \tilde{x}}{\partial a})^2$ is nonlinear; the term $\frac{dP}{d\rho}(\rho_0 \frac{\partial \tilde{x}}{\partial a})$ is also likely to be nonlinear, depending upon the appropriate equation of state. For some purposes, e.g., modeling the very intense acoustic waves that are produced by an explosion, this kind of nonlinear equation is necessary. But for many, everyday phenomena the nonlinear equation is not warranted and is an unnecessary encumbrance. For example, the condensation accompanying a mid-audio range sound having a frequency of 1000 Hz, and an amplitude that would be considered appreciable to the ear is very small, C is O(10^{-6}).[12] Thus the derivative $\partial \tilde{x}/\partial a$ is very, very close to 1 and the density that appears in $dP/d\rho$ can be taken to be the nominal density, ρ_0, with very little error. These approximations of small amplitude motion are often called the acoustic approximations, and with these in place the resulting governing equation is the linear, elementary wave equation,

$$\frac{\partial^2 \tilde{x}}{\partial t^2} = c^2 \frac{\partial^2 \tilde{x}}{\partial a^2}. \tag{17}$$

where the constant coefficient $c_0^2 = \frac{dP}{d\rho}(\rho_0)$ is the nondispersive wave speed squared. In this small amplitude (linear) limit there is nothing remaining that is characteristic of the Lagrangian system, and we might just as well have started with a simpler Eulerian representation.

The one-dimensional form of the Lagrangian equations (11), (15) and (16) are very handy, and if fluid dynamics was one-dimensional, then these equations would be our touchstone; we might never need anything more. However, one-dimensional phenomenon are exceedingly rare, and these one-dimensional equations do not admit, e.g., shear deformation, which two and three-dimensional fluids do with aplomb. Thus, in a one-dimensional flow the parcels maintain their relative order, while in a two- or three-dimensional flow, the trajectories are liable to become entangled[13] with no apparent order. This is evident even in the flow in a teacup, but not in the one-dimensional figures plotted here. Very commonly, a fluid flow will be fully three-dimensional and time-dependent, and a useful analysis will require the three-dimensional equations of motion. The three-dimensional Lagrangian equations are awkward

[12] D. H. Towne, *Wave Phenomenon*, Dover Pub., 1967, is highly recommended for more on acoustic phenomena.

[13] 'Entangled' is used here in the geometric and homey sense of a bowl of spaghetti (Fig. 4), and not in the quantum physics sense of entangled photons.

(difficult) for most purposes, and are not often used in their full form. If you are curious to see how or why this awkwardness arises, then read on. Otherwise you can skip ahead to Sec. 2.5 with little practical loss.

2.4 The agony of the three-dimensional Lagrangian equations

To see how the complication arises in three-dimension, begin with the component equations,

$$\frac{\partial^2 \tilde{x}}{\partial t^2} = -\frac{1}{\rho}\frac{\partial P}{\partial \tilde{x}}, \qquad \frac{\partial^2 \tilde{y}}{\partial t^2} = -\frac{1}{\rho}\frac{\partial P}{\partial \tilde{y}}, \qquad \frac{\partial^2 \tilde{z}}{\partial t^2} = -\frac{1}{\rho}\frac{\partial P}{\partial \tilde{z}} - g. \tag{18}$$

These are hopeful on first sight. But on second sight there is a significant problem with the pressure gradient, in that the spatial derivative is taken with respect to the material coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$, which in the Lagrangian system are *dependent* variables. In this form, the pressure gradient is the derivative of one unknown with respect to another unknown, which is extraordinarily difficult. The spatial derivative we would much prefer is with respect to an independent variable, which in the Lagrangian system is the initial position, the (a, b, c). Assuming that the transformation $\mathbf{A} \rightarrow \tilde{\mathbf{X}}$ exists, then the gradient may be written in terms of the (a, b, c). In general, the present position of a parcel will be dependent upon all three components of the initial position, i.e., $\tilde{x} = \tilde{x}(a, b, c)$, and vice versa, $a = a(\tilde{x}, \tilde{y}, \tilde{z})$. By the chain rule applied to composite functions, the $\tilde{x}-$component of the pressure gradient may be rewritten as

$$\frac{\partial P}{\partial \tilde{x}} = \frac{\partial P}{\partial a}\left(\frac{\partial \tilde{x}}{\partial a}\right)^{-1} + \frac{\partial P}{\partial b}\left(\frac{\partial \tilde{x}}{\partial b}\right)^{-1} + \frac{\partial P}{\partial c}\left(\frac{\partial \tilde{x}}{\partial c}\right)^{-1}, \tag{19}$$

and of course similarly for the $\tilde{y}$ and $\tilde{z}$ components (not written out). Using this form of the pressure gradient, the momentum equations are

$$\frac{\partial^2 \tilde{x}}{\partial t^2} = \frac{\partial P}{\partial a}\left(\frac{\partial \tilde{x}}{\partial a}\right)^{-1} + \frac{\partial P}{\partial b}\left(\frac{\partial \tilde{x}}{\partial b}\right)^{-1} + \frac{\partial P}{\partial c}\left(\frac{\partial \tilde{x}}{\partial c}\right)^{-1},$$

$$\frac{\partial^2 \tilde{y}}{\partial t^2} = \frac{\partial P}{\partial a}\left(\frac{\partial \tilde{y}}{\partial a}\right)^{-1} + \frac{\partial P}{\partial b}\left(\frac{\partial \tilde{y}}{\partial b}\right)^{-1} + \frac{\partial P}{\partial c}\left(\frac{\partial \tilde{y}}{\partial c}\right)^{-1},$$

and

$$\frac{\partial^2 \tilde{z}}{\partial t^2} = \frac{\partial P}{\partial a}\left(\frac{\partial \tilde{z}}{\partial a}\right)^{-1} + \frac{\partial P}{\partial b}\left(\frac{\partial \tilde{z}}{\partial b}\right)^{-1} + \frac{\partial P}{\partial c}\left(\frac{\partial \tilde{z}}{\partial c}\right)^{-1} - g. \tag{20}$$

The spatial derivatives of pressure are all with respect to the independent variables, the (a, b, c), which is progress, but the pressure gradient written in the material coordinates is quite formidable. It is possible to simplify the momentum equations somewhat by rearranging in a way that leaves only a single presure gradient term in each component equation, see Lamb,[9] Article 1.13.

$$\frac{\partial^2 \tilde{x}}{\partial t^2}\frac{\partial \tilde{x}}{\partial a} + \frac{\partial^2 \tilde{y}}{\partial t^2}\frac{\partial \tilde{y}}{\partial a} + (\frac{\partial^2 \tilde{z}}{\partial t^2} + g)\frac{\partial \tilde{z}}{\partial a} = -\frac{1}{\rho}\frac{\partial P}{\partial a},$$

$$\frac{\partial^2\tilde{x}}{\partial t^2}\frac{\partial\tilde{x}}{\partial b}+\frac{\partial^2\tilde{y}}{\partial t^2}\frac{\partial\tilde{y}}{\partial b}+(\frac{\partial^2\tilde{z}}{\partial t^2}+g)\frac{\partial\tilde{z}}{\partial b} = -\frac{1}{\rho}\frac{\partial P}{\partial b},$$

$$\frac{\partial^2\tilde{x}}{\partial t^2}\frac{\partial\tilde{x}}{\partial c}+\frac{\partial^2\tilde{y}}{\partial t^2}\frac{\partial\tilde{y}}{\partial c}+(\frac{\partial^2\tilde{z}}{\partial t^2}+g)\frac{\partial\tilde{z}}{\partial c} = -\frac{1}{\rho}\frac{\partial P}{\partial c}. \tag{21}$$

But now the accelertion terms are very difficult, and so this has evidently traded one problem, seen in (20), for another one that is no better. The bottom line is that we are not going to make direct use of these equations. Nevertheless there is value in knowing that the difficulty came from the pressure gradient, Eq. (19), and the dependence of parcel position upon the three-dimensional initial position, i.e., that $\tilde{x}=\tilde{x}(a,b,c)$. This suggests that a possible work-around could be to evaluate the pressure in field coordinates, i.e., the usual (x,y,z), and then apply the pressure gradient, computed in field coordinates at the position of the parcels, to each parcel. Thus, Lagrangian parcels can be tracked as they move and accelerate through an Eulerian grid. This is the strategy of the Particle in Cell model architecture discussed in Part 2.

The three-dimensional mass conservation equation is also complex but has a very clear interpretation,

$$\rho = \rho_0 \begin{vmatrix} \frac{\partial\tilde{x}}{\partial a} & \frac{\partial\tilde{x}}{\partial b} & \frac{\partial\tilde{x}}{\partial c} \\ \frac{\partial\tilde{y}}{\partial a} & \frac{\partial\tilde{y}}{\partial b} & \frac{\partial\tilde{y}}{\partial c} \\ \frac{\partial\tilde{z}}{\partial a} & \frac{\partial\tilde{z}}{\partial b} & \frac{\partial\tilde{z}}{\partial c} \end{vmatrix}^{-1} = \rho_0/J\left(\frac{\partial(\tilde{x},\tilde{y},\tilde{z})}{\partial(a,b,c)}\right), \tag{22}$$

where J is the determinant of the Jacobian of the transformation $\mathbf{A}\rightarrow\tilde{\mathbf{X}}$ (to be concise, often just the Jacobian, J). The Jacobian is the change in the unit volume when going from (a,b,c) to $(\tilde{x},\tilde{y},\tilde{z})$ (footnote 11). Notice that this density equation is diagnostic (no time derivative) rather than prognostic as are most budget equations. The density of this presumed Euler fluid may be calculated knowing the three-dimensional condensation (Sec. 2.3.1) given by the Jacobian above (which is asking quite a lot). If the fluid is incompressible, then $\rho=\rho_o$ and the Jacobian is unity and the mapping from $\mathbf{A}$ to $\tilde{\mathbf{X}}$ is volume conserving,

$$\frac{\rho}{\rho_o} = J\left(\frac{\partial(\tilde{x},\tilde{y},\tilde{z})}{\partial(a,b,c)}\right)^{-1} = 1 \tag{23}$$

(cf. the flow of Fig. 3 which has a small divergence and thus a small decrease in the number density of the material volume).

At the outset of this essay the question was posed: How can the conservation laws of classical physics be applied to the calculation of a fluid flow? Here in Eqs.(126), (21) and (22) is one possible answer for an ideal fluid; five equations in five unknowns, density ρ, the pressure p, and the parcel position, $(\tilde{x},\tilde{y},\tilde{z})$.[14] However, evaluating the pressure gradient terms is, not to put too fine a point on it, very difficult in a two- or three-dimensional Lagrangian system. With no intent to make this seem

[14] An insightful history of the Lagrangian equations and links to recent research is available from https://www.researchgate.net/profile/Uriel-Frisch/publication/260295078

hopeless, there are two hurdles presented by the Lagrangian system. (1) Some form of nonlinearity is inevitable in a theory of fluid mechanics, but the Lagrangian form of it, in which each term of the pressure gradient is multiplied by two or three deformation terms, is difficult to evaluate and to interpret. In some applications it would be appropriate to linearize these equations, but the result is (usually) the same linear equations that arise also from the Eulerian equations (Sec. 3). (2) The Lagrangian equations involve a second derivative with respect to time, where even one integration can be challenging. The (analytic) solution of the Lagrangian equations requires solving for parcel trajectories in one fell swoop. In contrast, within an Eulerian system we can solve for the velocity (one integration in time), and then integrate the velocity solution in a separate, offline step to find parcel trajectories (a theme of Part 2), if they are required. In many cases we may have no interest in knowing the trajectories of specific parcels and if so then a Lagrangian solution may tell us much more than we want to know. That would not be grounds for complaint, except that this additional information has to be provided, which is some part of the challenge for Lagrangian solutions. To summarize this subsection — it is a fair generalization[15] that the three-dimensional Lagrangian momentum equations, Eq. (21), are not as handy for most theoretical purposes as are the three-dimensional Eulerian equations that will be taken up in Sec. 3.

2.5 The joys of Lagrangian measurement

Lagrangian measurements come in many different forms, but all have in common that they seek to follow the position of a proxy of a fluid parcel, e.g., a neutrally buoyant float (Fig. 1) that moves more or less faithfully with the surrounding fluid. The identity of a specific float can be represented by a superscript index appended to the position vector, ${}^{j}\tilde{\mathbf{X}}(t)$. In practice, floats can be identified in the way they are tracked; the acoustically-tracked SOFAR floats of Fig. 1 transmitted within a twice-daily time window that was unique to each float. Satellite-tracked sea-surface drifters have an identifier that is transmitted via satellite link along with the drifter's position and the time. The basic, raw data of a Lagrangian float observation is thus a float identifier, a parameter that varies from one trajectory to another, the time, the independent variable, and the float position, the dependent variable. There will often be other measurements, including temperature, pressure and more, that may be encoded with the position reports.

2.5.1 An economy of scale

Lagrangian measurements can be advantageous for some purposes, and sometimes Lagrangian measurements are simply expedient, i.e., they may be the most practicle way to obtain extensive measurements. Lagrangian measurement methods may enjoy an economy of scale because it often

[15] To which there are many important exceptions, including adiabatic tracer advection problems considered here in Sec. 4. Just a few somewhat exotic examples: To the approximation of potential flow theory, a finite number of interacting, free vortices advect one another about in ways that can be remarkably intricate. Fig. 2 is the simplest example with just two vortices; imagine a collection of dozens of such vortices. There are also continuum problems wherein Lagrangian theory is tractable and even advantageous: in surface gravity wave problems, where the free surface is a constant in Lagrangian coordinates, and nonlinear acoustic waves where the full nonlinear equations may be reduced to a single governing equation, at least in one space dimension, Eq. (16). This list could go on and on, but for further examples in detail see instead the monograph by Bennett noted in footnote 10.

happens that the major cost of a float-tracking experiment lies in the tracking infrastructure, while additional floats or trackable particles are comparatively inexpensive and sometimes almost free.

Particle Imaging Velocimetry, PIV. A widely-used and rapidly developing observational technique involves the generation of the Eulerian velocity field by observing large numbers of (Lagrangian) floats or trackable particles. The basic idea is an application of the FPK: the trajectory of a given float, ${}^j\tilde{\mathbf{x}}(t)$, is time-differenced over a short time interval to give an estimate of the Eulerian velocity at the centered place and time of the Lagrangian data,

$$\mathbf{V}(\mathbf{X} = {}^j\tilde{\mathbf{X}}, t) \;=\; \frac{\partial\, {}^j\tilde{\mathbf{X}}(t)}{\partial t}. \tag{24}$$

This is a discretized version of the FPK Eq. (4) in that the tag ${}^j(\;)$ replaces the initial position $\mathbf{A}$ required in a continuum theory. The tag ${}^j(\;)$ may or may not be retained with the Eulerian velocity.

An implementation of this method on laboratory scale, known as Particle Imaging Velocimetry (PIV), was developed starting in the mid-1980s and is rapidly evolving and expanding today.[16] The PIV technique makes use of what can be very small and inexpensive particles that can be seeded into a laboratory flow in almost unlimited numbers, e.g., neutrally buoyant plastic beads in water. Assuming that the fluid is not opaque, the particles may be illuminated by a pulsed laser source and imaged with high speed cameras. Provided that the individual particles or the pattern that they form can be recognized from one image frame to the next, then it is fairly straightforward to differentiate particle or cluster position with respect to time, and so form a map, sometimes in three-dimensions and in great detail, of the Lagrangian fluid velocity throughout an experimental domain. If the temporal sampling interval is small enough that the particle's Lagrangian velocity is quasi-constant, then the estimated velocity makes a good estimate of the Eulerian velocity at the central time and position of the differentiation, and without the encumberance of a mechanical transducer. Tracers used in the PIV technique may be naturally occurring fluid properties, e.g., density or thermal anomalies[17]

The Global Drifter Program, GDP. The GDP is a world-wide consortium of oceanic, atmospheric and climate scientists that has deployed almost 30,000 instrumented buoys that drift freely on the ocean surface (Fig. 9).[18] The data stream includes the drifter identification, GPS-derived position, and sea

[16] A link to an extensive consortium of PIV investigators and applications is https://velocimetry.net/index.htm

[17] See http://www.irisa.fr/vista/Themes/Demos/MouvementFluide/infra.html The value of such detailed and comprehensive measurements is made especially clear by Crowley, C. J., J. L. Pughe-Sanford, W. Toler, M. C. Krygier, R. O. Grigoriev and M. F. Schatz, 2022, Turbulence tracks recurrent solutions, Proceedings of the National Academy of Sciences. DOI: 10.1073/pnas.2120665119 An application to biophysical problems in a natural environment is discussed by https://www.quantamagazine.org/what-can-jellyfish-teach-us-about-fluid-dynamics-20230628/

[18] These drifters consist of a positively buoyant surface float about 40 cm in diameter that holds batteries, sensors and satellite receivers and transmitters. A cable connects to a very large, slightly negatively buoyant drogue at 15 m depth that serves to tag the current at that near-surface depth. Drifters that lose their drogue display a spurious downwind motion that can be corrected. The basic drifter costs about $2000 and has a nominal life span of about one and a half years. To learn much more about drifters visit the GDP website, https://www.aoml.noaa.gov/phod/gdp. The vision of a collaborative, world-wide GDP is by Niiler, P. P., 2001: The world ocean surface circulation. In Ocean Circulation and Climate, G. Siedler, J. Church and J. Gould, eds., Academic Press, Volume 77 of International Geophysics Series, 193-204. For an insightful analysis and interpretation of surface drifter data see Laurindo, L. C., A. Mariano, and R. Lumpkin, 2017: An improved near-surface velocity climatology for the global ocean from drifter observations. Deep-Sea Res. I, 124, 73-92. The North Atlantic part of this analysis

Figure 9: The Global Drifter Program has observed the world ocean since the 1990s.

You could do this gig:
Drift freely on wind and waves,
tell us what you see.

surface temperature. Many drifters also measure barometric pressure and some also measure wind speed and direction. These data are transmitted in near-real time via satellite link to operational data centers and disseminated worldwide on the Global Telecommunication System (GTS). These near-realtime drifter data have considerable utility, including that they make a significant, cost-effective contribution to the body of marine observations available on the GTS network for the support of weather and marine forecasting.

A long-standing goal of the GDP is to make and report measurements over virtually the entire world ocean at roughly 500 km intervals. To this end, there are now (mid-November, 2022) roughly 1300 operational surface drifters contributed by twenty five partner nations. Drifters may be launched from almost any ship, and thereafter the sampling by these freely drifting instruments is inherently uncontrolled and so is liable to be nonuniform. The estimation of an Eulerian velocity field requires a mapping of the irregularly sampled drifter velocity estimates onto a preferred, regular grid, say via a kriging interpolation. Some form of time-averaging is usually applied at this stage in order to supress high frequency motions, such as tides. Thus the raw estimates from (24) are fairly heavily analyzed, which entails some risk of introducing artifacts.

Sampling may be correlated with the properties of the surface current. In the month June 2021 (Fig. (10) there were many drifters in the central subtropics, where there is a convergence of wind-diven, near-surface currents, (aka, the Sargasso Sea, after the floating macro-algae that also accumulates in this region) and there were very few drifters near the equator, which is a region of surface current divergence associated with easterly trade winds and a diverging near-surface current. If the winds vary seasonally, then there is the potential for a correlation between sampling density and the current, leading to a contribution to the estimated mean velocity sometimes called Stokes drift, about which more in Part 2, Sec. 2. The interpolated, time-averaged and wind-corrected near-surface velocity, Fig. (10), right, is said to be the pseudo-Eulerian mean velocity to acknowledge its considerable distance from the direct and simple FPK.

The large and growing GDP data base has been used to make detailed maps of the long term mean,

is shown in Fig. (10).

Eulerian near-surface velocity.[19] The North Atlantic basin is shown in Fig. (10), and since all currents are to the same scale, this emphasizes the largest velocities found near the western boundary of the basin and in the near-equatorial zone. The time-mean currents of the mid-ocean are very small by comparison, $O(10^{-2})$ of a western boundary current, and are not easy to discern here.[20] The western boundary current is generally northward from the equator to Labrador, and southward from the Denmark Strait to a confluence east of Labrador. Zonal near-equatorial currents occur in three bands, from north to south: eastward flow around 8 N, westward flow very near the equator, and westward flow also at about 8 S. The averaging into a (very) long-term mean as done here obscures that there is a substantial seasonal variation of the near-equatorial currents and the western boundary current within the Gulf of Mexico. The western boundary current otherwise shows little seasonal variation, i.e, the average summer looks a lot like the average winter.[21] The western boundary current is highly variable instantaneously (time scales of days to months), evident in the subsurface float tracks of Fig. (4).

2.5.2 Diagnosing source terms

If the goal of an experiment is to measure the force applied to a fluid, then by tracking parcels in time it is straightforward to estimate the acceleration, and go on to assert Newton's laws of classical dynamics in exactly the form used in classical (solid particle) dynamics, i.e.,

$$\frac{\partial^2 {}^{\mathbf{j}}\tilde{\mathbf{X}}}{\partial t^2} = \frac{1}{{}^{j}M}\mathbf{F}, \tag{25}$$

where $\mathbf{F}$ is the net force imposed upon that parcel by the environment, and M is the parcel mass. What that force may have been, i.e., whether a pressure gradient or frictional force, can not be told without supporting information.

The possible advantages of following the fluid (and thus a material volume) and directly observing fluid properties is illustrated by the tracer advection experiment of Fig. (3). When following the fluid, the tracer density changed only by virtue of divergence, there being no external source in that case. In many flows, the divergence is very small and may be taken as zero to a good approximation. In that event, the tracer is constant on moving parcels. Of course, to know where the resulting density applies will require observing the trajectory, but that is usually an integral part of any Lagrangian measurement. If observing in a fixed Eulerian frame, the tracer density changed by the same divergence plus a very large effect of advection of the tracer field through a control volume (advection in the sense $\mathbf{V}\cdot\nabla(\text{tracer})$ will be discussed in detail Sec. 3). Generally, the ocurrence of advection is expected, and is not necessarily interesting. On the other hand the possible source terms may require complex and uncertain parameterizations that need to be observed and checked. The comparative size of the source term and the advection term will vary greatly from one case to the next, and so no broad conclusions can be told. Nevertheless, if estimating a source term is an experimental goal, then Lagrangian measurements may offer some inherent advantage insofar as they account for the motion of the fluid via the directly observed

[19] https://www.aoml.noaa.gov/phod/gdp/ then search for 'mean velocity'.

[20] High resolution maps: https://www2.whoi.edu/staff/jprice/wp-content/uploads/sites/199/2022/12/NA-GDPmean-1.pdf

[21] Monthly maps: https://www2.whoi.edu/staff/jprice/wp-content/uploads/sites/199/2022/12/NA-monthly.mp4

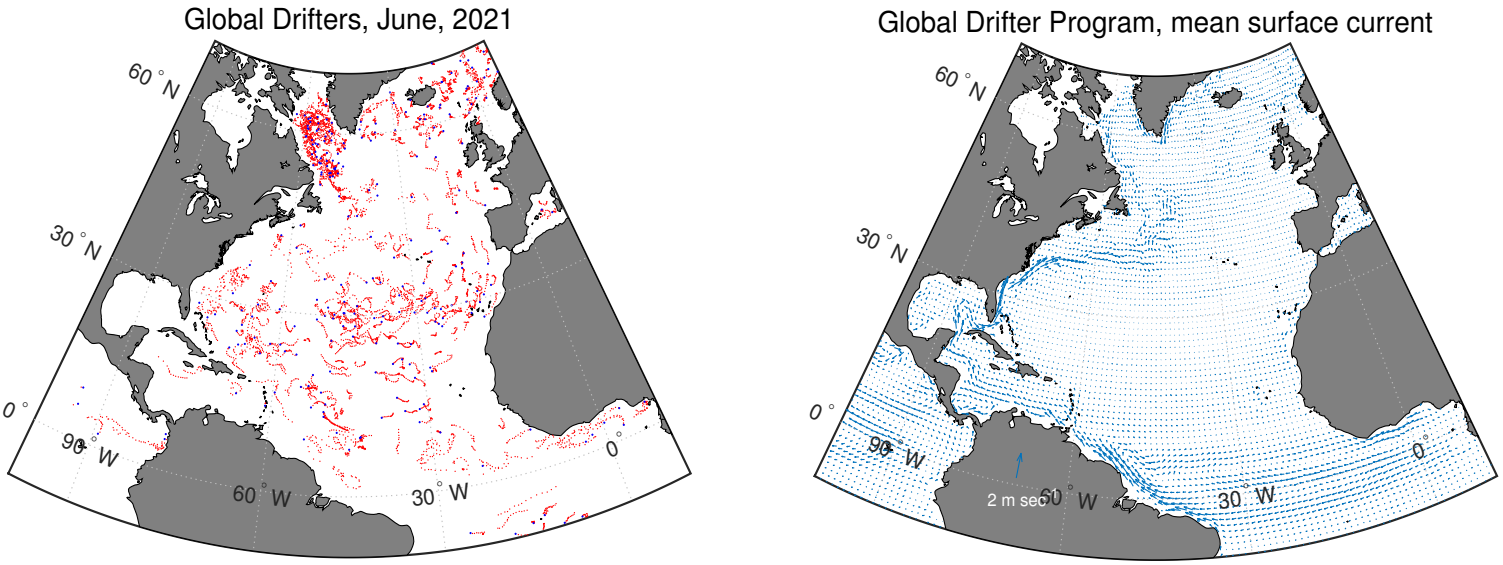

Figure 10: Surface drifter data acquired by the Global Drifter Program, GDP, and analyzed and archived by the Drifter Data Assembly Center, Atlantic Oceanographic and Meteorological Laboratory, National Oceanic and Atmospheric Administration.[18] (These figures are by the author.) **(left)** Drifter trajectories during June 2021. Red dots are at two day intervals; the last day is a blue dot. This is less than one percent of the total surface drifter data. During this month, the active drifters were distributed widely over the North Atlantic basin, but not uniformly. There were many drifters in the Labrador Sea following a recent, *ad hoc* deployment, and very few drifters in the Gulf of Mexico, which is somewhat blocked by the Windward Islands. **(right)** The time-mean velocity over three decades analyzed by Laurindo, Mariano and Lumpkin (2017).[18] The direct effect of the local wind was removed by an empirical fit. Notwithstanding the several analysis challenges, this is probably the best extant view of the time-mean, near-surface circulation of the oceans.

trajectory. An example of this is in Sec. 6, Part 2, where the float data of Fig. 1 will be analyzed for vorticity balance.

What's gone missing? This essay plus Part 2 emphasize two aspects of Lagrangian data analysis that are especially apropos the Lagrangian and Eulerian theme, *viz.*, large numbers of Lagrangian data facilitate mapping of velocity fields, Sec. 2.5.1, and, the inherently simpler tracer balance on observed trajectories can facilitate dynamical inferences, noted just above. This is very far from a complete account of the possible applications of Lagrangian data. Other important uses include inference of diffusion evident in the seemingly random dispersion of floats (as in Fig. 4)[22] and, the study of coherent Lagrangian structures, i.e., distinct flow features, typically large amplitude eddies, that retain their tracer content even while the surrounding fluid disperses tracer rapidly.[23]

[22] A highly recommended review of analysis methods for Lagrangian data is by LaCase, J. H., 2008: Statistics from Lagrangian observations, Prog. in Oceanogr., 77, 1-29. A discussion of the dispersion seen in Fig. (4) is https://www2.whoi.edu/staff/jprice/wp-content/uploads/sites/199/2024/01/Pdispers-WNA.pdf

[23] For these aspects of Lagrangian analysis, see Fredj, E., D. F. Carslon, Y. Amitai, A. Gozolchiani and H. Geldor, 2016, The particle tracking and analysis toolbox (PaTATO) for Matlab. Limnology and Oceanography Methds, 14, 586 - 599. doi:

2.6 Key ideas

1) A fluid flow can be viewed as an ongoing transformation of parcel positions from an initial position, $\mathbf{A}$, to a subsequent position $\tilde{\mathbf{X}}$. A sequence of positions of a given parcel (or float) is called a trajectory.

2) Under the plausible assumption that the determinant of the Jacobian of the transformation from $\mathbf{A}$ to $\tilde{\mathbf{X}}$ does not vanish, then trajectories $\tilde{\mathbf{X}}(\mathbf{A},t)$ may be inverted for $\mathbf{A}$.

3) The material time derivative $D/Dt = \partial/\partial t$ of parcel position is the Lagrangian velocity, and the second time derivative is the acceleration. These are the same as the solid particle velocity and acceleration.

4) The conservation laws (mass and momentum) apply to specific material volumes, and so in that important sense the conservation laws are what we would call Lagrangian. The equation of motion of a fluid parcel is the same as that of a solid particle. The issue for fluid mechanics is to know what the force is, since it comes from the interaction of a parcel with its complete surroundsings via normal and tangential stresses. (The ideal fluid model used to here considers only the former.)

5) In an ideal fluid, the density of a fluid parcel will be inversely proportional to the linear deformation (in one space dimension) or more generally to the Jacobian of the transformation $\mathbf{A} \rightarrow \tilde{\mathbf{X}}$ in two or three dimensions. A spatially uniform velocity will not cause a change in the volume or density of fluid parcels.

6) The Lagrangian representation of density comes in two parts: the density itself, $\tilde{\rho}(a,t)$, and the position (trajectory), $\tilde{x}(a,t)$. This is the basis for several useful soultion methods that will be discussed in Part 2.

7) The one dimensional Lagrangian conservation equations for mass and momentum may be reduced to an exact, nonlinear wave equation. One-dimensional flows are, however, a very limited and special case.

8) The three dimensional Lagrangian equations are quite complex, generally, and are not often used in their full form. The Lagrangian acceleration is not the problem; the pressure gradient is at issue, because the spatial gradient of pressure in a three-dimensonal flow will depend (in general) upon the three-dimensional initial position.

9) Lagrangian observation methods have the potential of two useful properties:

   i) Economy of scale. Lagrangian mesaurements may be feasible for very large numbers of floats or trackable particles, and so permit detailed mapping in space. Particle Imaging Velocimtery and the Global Drifter Program are examples on the laboratory and ocean basin scale.

   ii) Simpler dynamical inferences. The time rate change of a fluid property, say a tracer concentration, may be more readily attributed to a source, without having to estimate the often very

10.1002/lom3.10114. Also, Sebille, E., and colleagues, 2016, Lagrangian ocean analysis: Fundamentals and practices. Ocean Modelling: https://doi.org/10.1016/j.ocemod.2017.11.008

large effects of advection due to fluid motion (as is neccesary in an Eulerian system, Sec. 3). In effect, Lagrangian observations account for advection by observing directly the trajectory (a corollary of 6) above).

2.7 Problems

1) The discussion of fluid flow in this section omitted mention of diffusion in favor of an ideal fluid model that recognizes only resolved, larger-scale flow. Of course, diffusion actually happens, and is especially evident in smaller (tea cup-size) domains. In practice, how long can you follow a parcel (a small patch of dye) around in a stirred tea cup before it effectively disappears by diffusion into its surroundings? Does this depend upon the vigor or the details of the stirring?

2) The trajectories of Eq. (6) include the possibility of a spatially-uniform velocity, U_o. When evaluated in Fig. (8), U_o was set to zero. How would the solutions for density change if $U_o > 0$, say? Suppose instead that U_o is non-zero, but the a-dependent component = 0, i.e., that there is no linear deformation. How would the solutions for density change in that case?

3) Refer to the density solutions of Fig. (8). Can you use the Lagrangian solution to find the Eulerian solution, i.e., can you relate the left and right sides of Fig. (8)? This anticipates the Eulerian solutions of Sec. 3 and the method of characteristics in Sec. 4.

4) Can you show that the linear wave speed in the case of an adiabatic ideal gas is just $c_0^2 = \tilde{z}P_0/\rho_0$, and for the exact nonlinear wave equation, $c^2 = c_0^2 \frac{\partial \tilde{x}}{\partial a}^{-(\tilde{z}+1)}$. Thus the wave speed is greater in regions of condensation, $C > 1$, i.e., where the density is increased compared to ρ_0.

5) Can you spot the one-dimensional momentum equation (15) in the thicket of terms shown in Eqs. (21)? And, for bonus points, where is our friendly one-dimensional density equation (11) in the three-dimensional Jacobian?

6) Some fun with composite functions (reviewed in Sec. 6.2). Explain the difference between $\mathbf{V_L}(\mathbf{A},t)$ and $\mathbf{V_L}(\mathbf{A}(\tilde{\mathbf{x}},t),t)$.

3 The Eulerian (or field) coordinate system.

The goal of this section is to develop the Eulerian statements of mass and momentum conservation. The approach will be a start with the Lagrangian form of the kinematic relations and conservation laws, and then transform step by step to their Eulerian equivalent — velocity is in Sec. 3.1, time derivatives in Sec. 3.2, and volume integrals in Sec. 3.3.

3.1 The formal transformation of Lagrangian to Eulerian velocity

Let's start by using the FPK formalism to calculate the Eulerian velocity that corresponds to the (one-dimensional) Lagrangian trajectory and velocity of Eqs. (6) and (7). The parcel trajectories may be readily inverted to yield

$$a(\tilde{x},t) \;=\; \frac{\tilde{x} \;-\; U_o\, t}{(1 \;+\; 2t)^{-1/2}}.$$

Now eliminate a from $\tilde{u}$ of Eq. (7) in favor of the position, $\tilde{x}$, to find

$$\tilde{u}(a(\tilde{x},t),t) \;=\; U_o \;+\; \tilde{x}(1+2t)^{-1}.$$

This is the velocity at the position $x = \tilde{x}$ and time = t, which is just what we mean by Eulerian velocity, and so it is appropriate to relabel this function and the position $\tilde{x}$ to arrive at

$$u(x,t) \;=\; U_o \;+\; x(1+2t)^{-1}, \tag{26}$$

which is sampled in Fig. (6), lower. This procedure amounts to forming a composite function from the Lagrangian trajectory and velocity (composite functions are reviewed in Sec. 6.2). Notice that the spatially-uniform velocity, U_o, carries over directly.

The velocity field defined by Eq. (26) happens to look a lot like the Lagrangian velocity when plotted over the same range of a and x, cf. Figs. (6), middle and lower; u and $\tilde{u}$ are both positive and increase in the positive x direction. They both decrease with time. But while there are obvious similarities in this case, there are fundamental differences. The independent coordinates of u and $\tilde{u}$ are different things - the Lagrangian data depends upon a, the initial x-coordinate of a parcel, while the Eulerian data is a function of the field coordinate, x. To compare the Eulerian and the Lagrangian velocities is thus a bit like comparing apples and oranges; they are not the same kind of thing despite that they have the same dimensions, they describe the same flow, and in this very simple example, they do look a lot alike.

Though different generally, there are specific times and places where the Lagrangian and Eulerian velocities *are* equal, consistent with the FPK. To see this, let's follow the parcel tagged by $a = 0.5$, and choose an Eulerian observation site at (arbitrarily) $x = 0.7$. This parcel crosses the observation site at $t = 0.48$, Fig. (11) upper. At that time, and only at that time, the Lagrangian and Eulerian velocities are equal, cf., Fig. (11) middle. This is an exact equality, since there has been no need for approximation in this very simple transformation Lagrangian $\rightarrow$ Eulerian.

This transformation from the Lagrangian trajectories to the Eulerian velocity field required a time derivative of the trajectories, and the initial position was eliminated. Given only (26), we wouldn't know which parcel was present at a given x and time. In many cases this will not be of concern. But if the need was to know the origin of pollutants carried on the wind and observed at a field site, then it would be of first importance. To recover that Lagrangian information, and thus to go from an Eulerian velocity field to Lagrangian trajectories, requires an integration, taken up in Part 2.

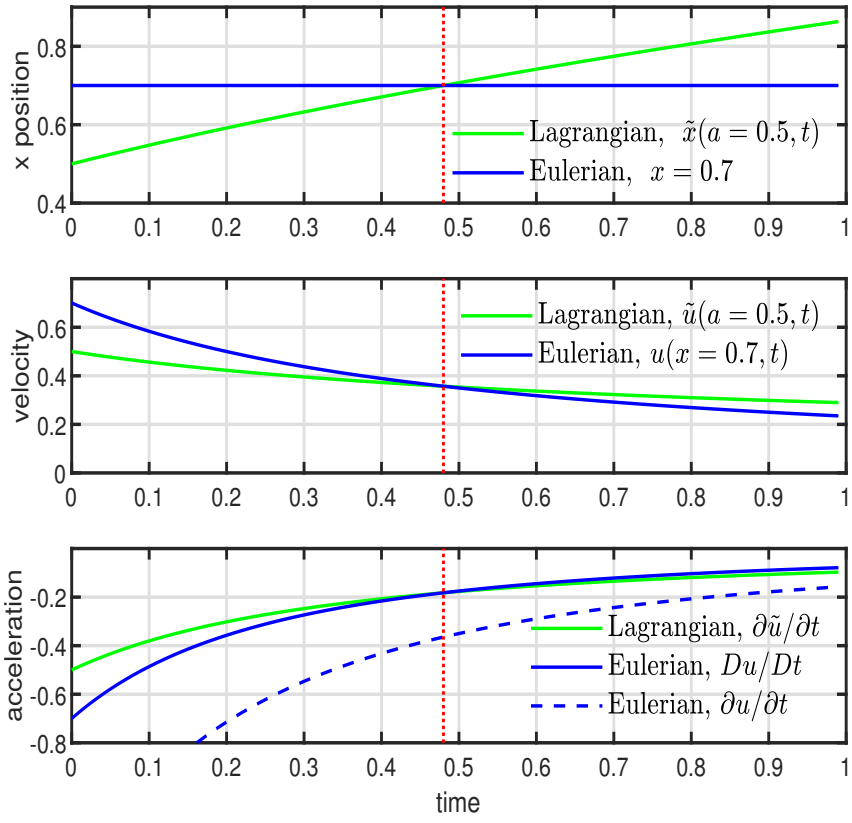

Figure 11: Lagrangian and Eulerian representations of the one-dimensional, time-dependent flow defined by the trajectories of Eq. (6) with $U_o = 0$. **(upper)** Positions. The trajectory of the parcel having $a = 0.5$ is the green, solid line. A fixed observation site, $x = 0.7$ is also shown (blue line). Note that this particular trajectory crosses $x = 0.7$ at time $t = 0.48$, computed from Eq. (6) and marked with a red dotted line in each panel. **(middle)** Velocities. The Lagrangian velocity of the parcel defined by $a = 0.5$ (green line) and the Eulerian velocity at the fixed position, $x = 0.7$ (blue line). Notice that the Lagrangian velocity of this specific parcel (green line) and the Eulerian velocity at the fixed position (blue line) are equal at $t = 0.48$, but not otherwise. This is illustrative of the FPK that was used to calculate the Eulerian velocity. **(lower)** Accelerations. The Lagrangian acceleration of the parcel $a = 0.5$ (green, solid line) and the Eulerian acceleration evaluated at the fixed position $x = 0.7$ (two blue lines). As will be discussed in Sec. 3.2, there are two ways to compute a time rate change of velocity in the Eulerian system; the partial time derivative holding x constant is shown as a blue dashed line, and the material time derivative, $Du/Dt = \partial u/\partial t + u\partial u/\partial x$, is a blue solid line. The latter is the counterpart of the Lagrangian acceleration $\partial\tilde{u}/\partial t$ in the sense that at the time the parcel crosses the Eulerian observation site, $Du/Dt = \partial\tilde{u}/\partial t$. Thus the Eulerian material derivative of velocity (the acceleration) at a fixed point and given time is equal to the Lagrangian acceleration of the parcel moving through that point at that time (but not otherwsie). This is the basis of the Eulerian equations of motion.

3.2 Transforming time derivatives; the material derivative

Transforming the acceleration, or time derivatives generally, is taken up next. Consider a scalar variable s in the Lagrangian system and so $\tilde{s}(a,t)$. The time rate of change holding a fixed is just

$$\frac{D\tilde{s}}{Dt} = \frac{\partial \tilde{s}}{\partial t}, \tag{27}$$

the partial time derivative, as noted before. In the case that $\tilde{s}$ is the fluid velocity given by Eq. (7), then

$$\frac{D\tilde{u}}{Dt} \;=\; \frac{\partial \tilde{u}}{\partial t} \;=\; -a(1+2t)^{-3/2}, \tag{28}$$

which is Eq. (8).

Now presume that the trajectory can be inverted so that $a = a(\tilde{x},t)$ is known, and hence so is $\tilde{s}(a(\tilde{x},t),t)$. The function of the inner variables alone is then $s(x,t)$, which is the Eulerian representation of s, or,

$$s(x,t) \;=\; \tilde{s}(a(\tilde{x},t),t). \tag{29}$$

Take the time derivative of the right hand side while holding a constant. By application of the chain rule for composite functions,

$$\frac{Ds(x,t)}{Dt} \;=\; \frac{\partial s}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial t} \;+\; \frac{\partial s}{\partial t}. \tag{30}$$

The time derivative $\partial\tilde{x}/\partial t$ holding a constant is just the velocity of a fluid parcel, $\tilde{u}$, and writing $\tilde{x}$ as x, then with minor rearrangement and relabelling, we arrive at

Eulerian material time derivative in 1-d: $$\boxed{\frac{Ds}{Dt} \;=\; \frac{\partial s}{\partial t} \;+\; u\frac{\partial s}{\partial x}} \tag{31}$$

This may be exercised and checked by transforming the material derivative of the Eulerian velocity field $u(x,t)$ of Eq. (26) into the corresponding Lagrangian form to compare with the Lagrangian acceleration given by Eq. 28. The material time derivative of this velocity field is just

$$\frac{Du}{Dt} \;=\; (\frac{\partial}{\partial t} \;+\; u\frac{\partial}{\partial x})x(1+2t)^{-1} \;=\; -2x(1+2t)^{-2} \;+\; x(1+2t)^{-2} \;=\; -x(1+2t)^{-2}.$$

To find the corresponding Lagrangian acceleration, eliminate this x in favor of a using the known trajectory for this flow, Eq. (6), to find

$$\frac{Du}{Dt}(\tilde{x}(a,t),t) \;=\; -a(1 \;+\; 2t)^{-3/2},$$

which is the same as the Lagrangian acceleration, Eq. (28). Thus, the Du/Dt given by Eq. (31) is indeed the Eulerian equivalent of $D\tilde{u}/Dt$, as intended.

In a three-dimensional Cartesian system with coordinates (x,y,z) and velocity (u,v,w), the material

time derivative represented in an Eulerian system is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \tag{32}$$

or using more compact vector notation,[24]

$$\text{Eulerian material derivative in 3-d:} \qquad \boxed{\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + V\cdot\nabla(\ )} \tag{33}$$

The importance of the D/Dt operator can by gauged from the veritable boatload of names that it has acquired in the literature — D/Dt is called the convective derivative, the substantive or substantial derivative, the Stokes derivative and the material derivative (the choice here). D/Dt is also said to be the time derivative 'following the flow'. This is appropriate insofar as $\partial\tilde{x}/\partial t$ in Eq. (30) is with a held constant. However, this could be misleading if it was interpreted to mean parcel tracking in the Lagrangian sense, i.e., for an extended time. Rather, the operator D/Dt gives, entirely in field coordinates (just x in the one-dimensional case), the time rate of change that would be observed by a parcel moving through the point x at the time t; thus D/Dt follows the flow only instantaneously in both x and t. An example is in Fig. 11, lower; the Lagrangian acceleration and the material time derivative of velocity (evaluated in their respective systems) are equal when they are evaluated at a common position and time, in Fig. (11) at $a = 0.5$, $x = 0.7$ and t= 0.48, in the same way that the corresponding Lagrangian and Eulerian velocities are equal at this common position and time, Fig. (11), middle.

3.3 Transforming integrals and their time derivatives via the Reynolds Transport Theorem

The conservation laws of classical physics apply to material volumes, e.g., the moving gray volume of Fig.(3), and not to control volumes that are fixed in space. The Eulerian equivalent of a moving, material fluid volume will be the third key piece of the transformation of kinematics and dynamics from Lagrangian to Eulerian form. This relation is called the Reynolds Transport Theorem, or RTT (and yes, this is the same Reynolds of Reynolds number fame in Sec. 6.1.2).

[24] The gradient operator expanded in Cartesian coordinates is

$$\nabla = e_x\frac{\partial}{\partial x} + e_y\frac{\partial}{\partial y} + e_z\frac{\partial}{\partial z},$$

with **e** the unit vector. The advective derivative term $V\cdot\nabla$ of (33) is, in effect, a scalar that is multiplied onto the variable being differentiated, and it is sometimes written $(V\cdot\nabla)$ to emphasize this property. In a Cartesian coordinate system the D/Dt operator produces four terms for each component of a vector being differentiated, one term being the partial with respect to time, and three terms arising from the advective rate of change. However, when D/Dt is expanded in cylindrical polar coordinates it becomes five terms, the additional advective term arising from the θ dependence of the r and θ unit vectors (in the Cartesian system the unit vectors are constant). In spherical polar coordinates the D/Dt operator expands to six terms. It is sometimes helpful to represent the advective derivative of velocity by the following vector identity $(\mathbf{V}\cdot\nabla)\mathbf{V} = \frac{1}{2}\nabla(\mathbf{V}^2) + (\nabla\times\mathbf{V})\times\mathbf{V}$, which is less likely to be misinterpreted. This form is especially useful when treating an ideal fluid, wherein frictional effects vanish and so too may the curl of the velocity, also called the vorticity (Sec. 4).

The essential content of the RTT can be understood by considering the integral in R^1 of an intensive fluid property, say, $\tilde{s}$, over a moving material volume of fluid (Fig. 7) (this is, of course, a linear volume, the results being easily extended to R^3),

$$\tilde{S}(t) \;=\; \int_{\tilde{x}_1}^{\tilde{x}_2} \tilde{s}(\tilde{x},t)d\tilde{x}. \tag{34}$$

The volume integral $\tilde{S}(t)$ is an extensive property, which may be subject to a physical conservation law,[25] say $\frac{DS}{Dt} = 0$, or if there is a source Q then, $\frac{DS}{Dt} = Q$, (examples will follow). To make the best use of such a conservation law we need to write the integral Eq. (34) in terms of the field variables, just x in one dimension.

To proceed, represent the integral by a sum over some large number K of infinitesimal segments, $\delta\tilde{x}_k$,

$$\tilde{x}_2 - \tilde{x}_1 \;=\; \sum_K \delta\tilde{x}_k,$$

$$\tilde{S}(t) \;=\; \int_{\tilde{x}_1}^{\tilde{x}_2} s(\tilde{x},t)d\tilde{x} \;\approx\; \sum_K s(\tilde{x}_k,t)\delta\tilde{x}_k. \tag{35}$$

The number K is fixed while the length of the total interval, $\tilde{x}_2 - \tilde{x}_1$, may change with time and thus we have to allow that the material segments $\delta\tilde{x}_k$ will vary with time. With that understood, the time derivative can be taken through the summation operator,

$$\frac{D\tilde{S}}{Dt} \;\approx\; \sum_K \left(\frac{D\tilde{s}(\tilde{x}_k)}{Dt}\delta\tilde{x}_k \;+\; \tilde{s}(\tilde{x}_k)\frac{D\delta\tilde{x}_k}{Dt}\right). \tag{36}$$

The first term on the right hand side is expected, and the second term accounts for a possible change in the material length, $D\delta\tilde{x}_k/Dt$. The time derivative of this length is just the velocity difference at the location of the two parcels that mark its endpoints,

$$\frac{D\delta\tilde{x}_k}{Dt} \;=\; \frac{D\delta x_k}{Dt} \;=\; \frac{D}{Dt}(x_{k+1} \;-\; x_k) \;=\; u_{k+1} - u_k,$$

where $u_k \;=\; \frac{Dx_k}{Dt}$ is the fluid velocity at $x_k = \tilde{x}_k$, etc. When the length δx is infinitesimal the velocity difference may be written

$$(u_{k+1} - u_k) \;=\; \frac{\partial u}{\partial x}\delta x$$

[25] It is sometimes helpful to make clear whether a given fluid property is 'intensive' or 'extensive'. (1) An intensive property of a fluid is measurable at a point, e.g., density, while an extensive property of a fluid is an integral over a finite volume, e.g., the volume itself or the mass of the fluid volume. An extensive property is thus probably not directly measurable. (2) Imagine two volumes of fluid, V_1 and V_2, having densities ρ_1 and ρ_2. Now suppose that the volumes are added together to make a new volume, V_3. Aside from from nonlinear effects in the equation of state, the new volume will be $V_3 = V_1 + V_2$, while the new density will be $\rho_3 = (\rho_1 V_1 + \rho_2 V_2)/V_3$. Thus the extensive property, the masses, add up when volumes are combined, while the corresponding intensive property, the densities, are a volume-weighted average. (3) Extensive properties such as mass, and internal energy, may be subject to a conservation law, while the corresponding intensive properties, density and temperature, will generally not be. Nevertheless, as we will see in this section, conservation laws for extensive properties will lead to useful differential balance equations for the intensive properties.

and hence

$$\frac{D\delta x}{Dt} = \frac{\partial u}{\partial x}\delta x.$$

Collecting these results into Eq. (36) gives

$$\frac{D\tilde{S}}{Dt} = \approx \int_{x_1}^{x_2}(\frac{Ds}{Dt} + s\frac{\partial u}{\partial x})dx. \tag{37}$$

In one dimension, the time derivative over a material volume is related to the equivalent field quantities via the important

Reynolds Transport Theorem in 1-d:
$$\boxed{\frac{D\tilde{S}}{Dt} = \frac{D}{Dt}\int_{\tilde{x}_1}^{\tilde{x}_2}\tilde{s}(\tilde{x},t)\,d\tilde{x} = \int_{x_1}^{x_2}(\frac{Ds}{Dt} + s\frac{\partial u}{\partial x})\,dx} \tag{38}$$

where $x_1 = \tilde{x}_1$, etc. at the time the transformation is made. Thus to transform the time derivative of a material volume into field coordinates requires accounting for the time rate of change of volume. (We will demonstrate the RTT in an application to mass conservation discussed below.)

Generalization of the RTT to three-dimensions: If instead of a one-dimensional 'volume' used in the examples above, suppose that the issue is a three-dimensional volume, $dVol = dx\,dy\,dz$ in Cartesian coordinates. It is straightforward to show that the time derivative of the differential volume is

$$\frac{D\ dVol}{Dt} = (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})dVol = \nabla\cdot V\ dVol \tag{39}$$

$\nabla\cdot V$ is the divergence of the fluid velocity in three dimensions and is the volumetric (or normalized) time rate change of the differential volume, i.e., $\nabla\cdot V = dVol^{-1}(d/dt)(dVol)$. The RTT in three dimensions is then a straightforward generalization of the one-dimensional form, Eq. (38),

RTT in 3-d:
$$\boxed{\frac{D}{Dt}\iiint_{material}\tilde{s}\,dVol = \iiint_{field}(\frac{Ds}{Dt} + s\nabla\cdot V)\,dVol} \tag{40}$$

The RTT is an exact, kinematic relationship that holds for an intensive fluid property, e.g., s could be the mass density, ρ, or the momentum density, $\rho\mathbf{V}$. The RTT serves a crucial purpose — to show how fluid flow effects the time rate change of the corresponding extensive quantity, i.e., the S in our notation, and moreover, to show this crucial result in field coordinates. The RTT is thus the last piece needed to give a practical (widely useful) answer to the question posed in the opening paragraph of Section 1, *viz.*, how can we apply the conservation principles of classical physics to a fluid flow? The statement of mass and momentum are discussed below. Physics, as opposed to kinematics, enters the development when the corresponding extensive quantity, e.g., mass or momentum, is asserted to be conserved under fluid flow, and including external sources, if any.[26]

[26] A superb reference on the Reynolds Transport Theorem and much else that is relevant to fluid dynamics is by C. C. Lin and L. A. Segel, *Mathematics Applied to Deterministic Problems in the Natural Sciences* (MacMillan Pub., 1974).

3.4 The Eulerian equations of motion

3.4.1 Mass conservation represented in field coordinates

An important application of the RTT is to the mass of a moving, three-dimensional volume of fluid, which is thus an extensive property defined on a specific material volume. There is no source or sink for mass in the classical physics that is presumed to hold, and thus the physics of mass conservation is that the mass $\tilde{M}$ of a specific material volume (and thus the tilde) must remain exactly constant,

$$\frac{D\tilde{M}}{Dt} = 0, \tag{41}$$

for all flow conditions. We can make no such assertion for the corresponding intensive property, the mass density, ρ, nor for the mass inside of a control volume that is fixed in space.

The mass conservation Eq. (41) has a clear and precise physical meaning, but it is not in and of itself directly useful as a means to predict mass or density in most models of fluid flow. A form that will be useful follows from application of the Reynolds Transport Theorem,

$$\frac{D\tilde{M}}{Dt} = \frac{D}{Dt}\iiint_{material} \tilde{\rho}\, dVol = \iiint_{field} \left(\frac{D\rho}{Dt} + \rho\nabla\cdot V\right) dVol = 0. \tag{42}$$

If this integral relation holds at all times and for all positions within a domain, and if the integrand is smooth (no discontinuities), then the integrand must vanish at all times and positions in that domain[27] and so yielding the differential form of

$$\text{Eulerian mass conservation:} \quad \boxed{\frac{D\rho}{Dt} + \rho\nabla\cdot V = 0} \tag{43}$$

The meaning of Eq. (43) is as follows: say the material derivative of density is negative, $\frac{D\rho}{Dt} < 0$ at a point in space, then given that $\rho > 0$, there must be a divergence of the fluid velocity, $\nabla\cdot\mathbf{V} > 0$, at that point. Thus the fluid parcel that is instantaneously at that point will have decreasing density and an inversely proportional increasing volume. This is a kinematic relationship that holds regardless of what the cause of the density change might be, i.e., whether due to a pressure variation or heat exchange with the surroundings, or even a phase change of the fluid material. Thus Eq. (43) is not by itself sufficient to predict or understand *why* density might change, which comes instead from a thermodynamic state equation, as discussed further below. However, it is worth noting that Eq. (43) does *not* hold if the fluid volume exchanges material with its surroundings by diffusion, e.g., say of salt in the case of sea water or water vapor if air. In that (very common) case we would have to allow a source term appropriate to the

[27] The Dubois-Reymond lemma: it is given that an integral of the sort Eq. (42) but here in R^1, $\int_{x_1}^{x_2}\sigma(x)dx = 0$, vanishes for any x_1, x_2 and that σ is smooth. If we now suppose that $\sigma(x_a) > 0$ this leads to a contradiction. Given that σ is smooth, then there will be some neighborhood around x_a where $\sigma(x) > 0$. Choose x_1 and x_2 to lie within this neighborhood, and apply the mean value theorem to the integral to find that $\sigma(\bar{x})(x_2 - x_1) \geq 0$. This contradicts what is known about this integral, and hence the conclusion is that $\sigma(x)$ must be zero at every point. In other words, the only smooth function whose integral is zero on every interval is the zero function.

specific conditions.

The mass conservation relation may be written in a form that shows the separate effects of advection and of velocity divergence; in one dimension,

$$\frac{\partial \rho}{\partial t} \;=\; -\,(\,u\frac{\partial \rho}{\partial x} \;+\rho\frac{\partial u}{\partial x}), \tag{44}$$

and for two or three-d,

$$\frac{\partial \rho}{\partial t} \;=\; -\,(\mathbf{V}\cdot\nabla\rho \;+\; \rho\nabla\cdot\mathbf{V}). \tag{45}$$

The one-dimensional density and velocity of Sec. 2.2 provide an example; recall that the Lagrangian density and the trajectory were, repeating Eqs. (6) and (14),

$$\tilde{\rho}(a,t) \;=\; \frac{\rho_c+\Gamma a}{(1+2t)^{1/2}} \quad \text{and} \quad \tilde{x}(a,t) \;=\; U_o t \;+\; a(1+2t)^{1/2}, \tag{46}$$

from which the Eulerian density may be calculated by algebra; eliminate a in favor of $x \;=\; \tilde{x}$ to find,

$$\rho(x,t) \;=\; \frac{\rho_c}{(1+2t)^{1/2}} \;+\; \frac{\Gamma(x-U_o t)}{(1+2t)}, \tag{47}$$

(Fig. (8), right) and recall the Eulerian velocity,

$$u(x,t) \;=\; U_o + \frac{x}{(1+2t)}. \tag{48}$$

To exercise the Eulerian density equation (44), compute the local rate of change of this density

$$\frac{\partial \rho}{\partial t} \;=\; -\,(\frac{\rho_c}{(1+2t)^{3/2}} \;+\; \frac{2\Gamma(x-U_o t)}{(1+2t)^2} \;+\; \frac{\Gamma U_o}{1+2t}), \tag{49}$$

and verify that it is the sum of an advection term, evaluated from (47) and (48),

$$u\frac{\partial \rho}{\partial x} \;=\; \frac{\Gamma(x-U_o t)}{(1+2t)^2} \;+\; \frac{\Gamma U_o}{(1+2t)}, \tag{50}$$

and a divergence term,

$$\rho\frac{\partial u}{\partial x} \;=\; \frac{\rho_c}{(1+2t)^{3/2}} \;+\; \frac{\Gamma(x-U_o t)}{(1+2t)^2}. \tag{51}$$

The divergence term (51) is negative (when on the right hand side of Eq. 44) since this flow is divergent, $\frac{\partial u}{\partial x} > 0$, and fluid volumes are stretched by the flow. This divergence process is in common with the Lagrangian density equation, (11), where it appears as the linear deformation, effectively the time-integrated divergence. The advection term is also negative, because $u > 0$, and the density increases toward positive x. The effect of translating or 'advecting' this density profile toward positive x is thus to cause the density observed at a fixed point to decrease with time.

It is important appreciate that the density observed at a fixed point can change due solely to an

advection process even while the density observed on moving parcels may remain constant. The only thing required in one dimension is a spatial variation of density combined with a nonzero velocity, and that can include the spatially-uniform velocity, U_o. In a more realistic multi-dimensional space, the condition is that $\mathbf{V}\cdot\nabla\rho$ is nonzero. A 'frozen field' balance is said to hold when the local rate of change is due mainly to advection, and is not uncommon.

Now that we have a Lagrangian and Eulerian solution side by side in (46) and (47) it is interesting and worthwhile to consider how they are different, despite that they describe the same 'flow'. The Eulerian solution (47) is complete in one piece, $\rho(x,t)$ (assuming the x-coordinates are known, of course). The Eulerian solution contains within it the effects of divergence of the fluid motion, as well as the effects of the fluid motion, the advection process. Compare this with the Lagrangian solution (46) that comes in two pieces, the density on moving parcels $\tilde{\rho}(a,t)$, which in this presumed ideal fluid model changes only by divergence (linear deformation), and the time-dependent position of those parcels, $\tilde{x}(a,t)$. The latter is the fluid motion. The two-part Lagrangian solution thus simplifies the density equation considerably, an idea that can be exploited to real advantage in the method of characteristics and in various numerical schemes, e.g., semi-Lagrangian advection, both taken up in Part 2. But, you have to know the trajectory.

3.4.2 The flux form of the Eulerian equations.

Another way to write the Eulerian mass conservation equation (43) is to collect terms under the gradient operator,

$$\text{mass conservation, flux form:} \quad \boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = 0} \tag{52}$$

An Eulerian budget equation in this form is said to be in 'flux' or 'conservation' form and has two important properties. To make it clear that these properties follow from the conservation form of the equation rather than the physical properties of the intensive property mass density, ρ, it is helpful to consider the same equation for an arbitrary, intensive scalar s,

$$\frac{\partial s}{\partial t} + \nabla\cdot\mathbf{s} = 0 \tag{53}$$

which could be mass per unit volume, ρ, or momentum per unit volume, $\rho\mathbf{V}$. Thus $s\times Volume = S$ is an extensive property, e.g., mass or momentum. The vector $\mathbf{s}$ is the flux of s and could be due to transport by the moving fluid, e.g., $\mathbf{s} = s\mathbf{V}$ obtains for any fluid property. The flux could also arise from molecular diffusion, $\mathbf{s} = -K\nabla s$, if a Fickian diffusion law is appropriate. In either event, $\mathbf{s}$ is a vector with dimensions $s \doteq s\ length\ \ time^{-1}$ so that $s\times area \doteq S\,time^{-1}$. For the moment it will be assumed that $\mathbf{s}$ depends linearly upon s, so that $s = 0$ over some neighborhood implies that the flux $\mathbf{s} = 0$ as well.

A volume integral of s taken over a fixed portion of a domain, i.e., a control volume (Fig. 13), gives the total amount of s within the control volume,

$$S = \int_{x1}^{x2}\int_{y1}^{y2}\int_{z1}^{z2} s\,dxdydz = \iiint_{field} s\,dVol.$$

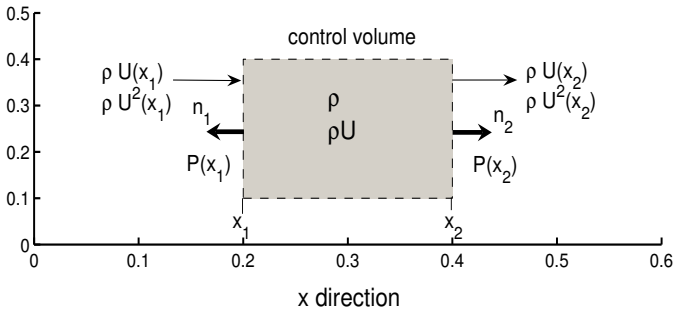

Figure 12: A schematic of a control volume (fixed in space) set in an ideal fluid in order to evaluate the Eulerian budgets of mass and momentum. The face of the control volume that is normal to the x-direction has an area A (not shown). In this one-dimensional Eulerian system the independent coordinates are the x-position, and the time, t. The dependent variables are the fluid density, $\rho(x,t)$, the momentum, $\rho u(x,t)$, and the pressure, $P(x,t)$. The one-dimensional fluxes of mass, ρu, and momentum, ρu^2, are implied by the arrows coming into or leaving the control volume.

This volume integral taken over the left and right sides of Eq. (53) gives

$$\iiint\limits_{field} \frac{\partial s}{\partial t}\, dVol \;=\; -\iiint\limits_{field} \nabla\cdot\mathbf{s}\, dVol \tag{54}$$

The time derivative can be moved outside the volume integral over the fixed limits, x_1, x_2, etc., of the control volume and Gauss' divergence theorem used on the right side to find that

$$\frac{\partial}{\partial t}\iiint\limits_{field} s\, dVol \;=\; \frac{\partial S}{\partial t} \;=\; -\iint\limits_{field} \mathbf{s}\cdot\mathbf{n}\, da. \tag{55}$$

Thus S can change only because of a flux across the surface of the control volume, Figs. (12) and (13). If it happens that the flux is uniform in space and so is non-divergent, $\nabla\cdot\mathbf{s} = 0$, Eq. (54), then S will remain constant in time. The first key point is that the quantity of interest for Eulerian budgets is the *flux divergence*, rather than the flux itself.

Now imagine that the boundaries of the control volume expand to infinity. It may be assumed without loss of generality that the variable b and the flux $\mathbf{s}$ will be vanishing at infinity, i.e., that the flow is bounded in space, and hence the flux on the surfaces vanishes as well. Over this control volume it follws that

$$\iint \mathbf{s}\cdot\mathbf{n}\, da \;=\; 0 \;=\; \frac{\partial S}{\partial t}$$

and S is constant in time. The second key point is that, given the present boundary conditions, the total amount of s can not change due to the flux $\mathbf{s}$, even though the flux $\mathbf{s}$ may act to redistribute b within the domain so that at a given point in space, s could be highly time-dependent. The same thing would result if the flux $\mathbf{s}$ vanished on the boundary of a finite domain, e.g., if the flux was due to fluid motion and the domain boundary was an impermeable surface.

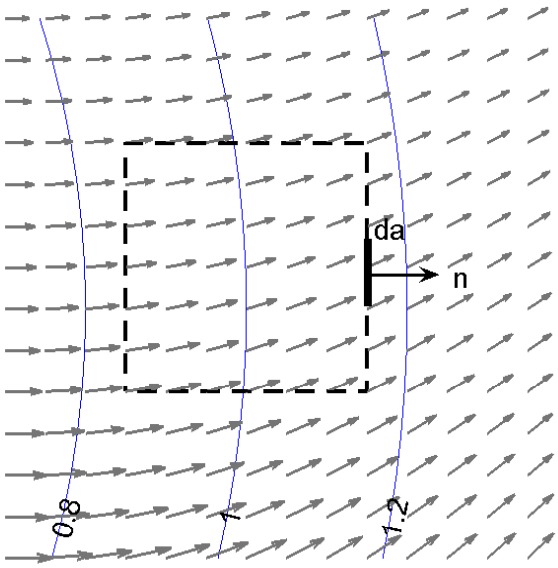

Figure 13: A control volume is fixed in space and has imaginary sides (dashed lines) and a two-dimensional slice through a velocity field, $\mathbf{V}$, and isolines of a fluid intensive property, s. The flux of b due to transport by the fluid velocity is $\mathbf{s} = s\mathbf{V}$; the flux of s through a differential surface element of the control volume is then $\mathbf{s} \cdot \mathbf{n} da$, where $\mathbf{n}$ is the outward unit vector normal to the surface. A question for you: given what may be seen from this figure, how will the total amount of s in this control volume, S, change with time?

Incompressible flow: for many purposes, fluids may appear to be infinitely stiff. The velocity divergence term is clearly necessary in the mass conservation equation, and plays a prominent role in the one-dimensional example. Nevertheless, for most phenomenon of the atmosphere or ocean and for many engineering flows, the fluid velocity that is associated with the velocity divergence is very, very small compared to the (actual) fluid velocities and may often be ignored with no appreciable error. Under this incompressibility assumption, the velocity is assumed to follow

$$\text{incompressible flow:} \quad \boxed{\nabla \cdot \mathbf{V} = 0} \tag{56}$$

which in effect says that the volume (rather than the mass) of fluid parcels is constant in time. In one-dimension this requires the velocity to be spatially uniform, which is not an interesting case (and so our one-dimensional flow is highly and unrealistically divergent). In two dimensions or three dimensions this need not be the case. For example, in two dimensions, a gentle linear deformation in say the x-direction may be almost exactly compensated by a linear deformation of the opposite sign in the y-direction, with the result that the area of a two-dimensional material volume will remain nearly constant and so the fluid will appear to be very nearly incompressible, even while the shape of the material volume may change greatly.

Even with the incompressibility assumption in place, it is not inconsistent to have a model in which density may nevertheless change, and indeed density changes may be of primary importance in causing the flow of an incompressible fluid. Under the incompressibility assumption, density may be computed from an equation of state given the pressure, temperature, salinity, etc., with no reference made to the divergence of the fluid velocity. That the divergence may be ignored is a useful mathematical

approximation that is contingent upon the physical phenomenon under consideration. One important class of phenomena, acoustic waves, owe their entire existence to velocity divergence and associated pressure changes. The incompressibility assumption is valid provided that the fluid velocity is much less than the speed of sound, which holds well for most natural flows of the atmosphere and ocean.

3.4.3 Momentum conservation represented in field coordinates

A second and very important application of the Reynolds Transport Theorem arises on consideration of momentum balance. The momentum of a moving, three-dimensional volume of fluid can be written

$$\tilde{N} = \iiint\limits_{material} \tilde{\rho}\tilde{V}\, dVol. \tag{57}$$

Because this is a material volume we can assert Newton's Second Law, that the momentum of this specific volume of fluid can change only if there is a net force, F,

$$\frac{D\tilde{N}}{Dt} = \iiint\limits_{material} F\, dVol, \tag{58}$$

acting on the fluid. This $\tilde{\mathbf{F}}$ could include inertial forces, such as gravitational mass attraction or the Coriolis force that act throughout the volume, and normal or tangential stresses (Section 6.1) that act on the surface of the volume. By means of the Reynolds Transport Theorem and the mass conservation relation, the left side of Eq. (58) may be written in field coordinates as

$$\frac{D\tilde{N}}{Dt} = \frac{D}{Dt}\iiint\limits_{material} \tilde{\rho}\tilde{\mathbf{V}}\, dVol \;=\; \iiint\limits_{field} \rho\frac{DV}{Dt}\, dVol. \tag{59}$$

A term that might have been expected,

$$\frac{D\rho}{Dt}V, \tag{60}$$

has dropped out by application of the mass conservation requirement. Thus the momentum of a fluid parcel (or marked fluid volume) can change only because of a change of the velocity. The right side of Eq. (58) is a volume integral over field coordinates that is evaluated over the volume occupied instantaneously by the moving fluid,

$$\iiint\limits_{material} \tilde{F}\, dVol \;=\; \iiint\limits_{field} F\, dVol, \tag{61}$$

and thus from Eqs. (59) and (61),

$$\iiint\limits_{field} \rho\frac{DV}{Dt}\, dVol \;=\; \iiint\limits_{field} F\, dVol. \tag{62}$$

The volume considered here is arbitrary, and so the integral relation holds for every volume. The differential form of the momentum balance for a fluid continuum is

$$\text{Eulerian momentum balance:} \quad \boxed{\frac{DV}{Dt} = \frac{\partial V}{\partial t} + (V\cdot\nabla)V = F/\rho} \tag{63}$$

The volume of the integrals in Eq. (62) is the volume that is instantaneously coincident with the material volume indicated in Eq. (57). For the purpose of finding F, this may be regarded as a fixed control volume, and the stress acts upon the fluid that is instantaneously contained within the control volume.

3.4.4 Fluid mechanics requires a stress tensor

Now to specify the force, F, which will proceed in two steps. The first step amounts to mathematical bookkeeping on the stress tensor, essentially a 3x3 matrix of stress components that may be used to evaluate the stress vector acting on any face of a control volume. The second step is to specify what those stress components are. Stresses within a fluid depend upon the physical properties of the fluid and the flow itself, and so this involves something more than kinematics and bookkeeping.

The specification of the stress on a given surface requires a 3x3 object because the surface and the stress vector are both 3-dimensional (Fig. 19). That is, the stress vector is defined by three components, and the surface upon which the stress acts is defined by the unit normal vector, and thus another three components. A full representation of the stress within a fluid generally requires nine components at each point in space. It can be very helpful to think of these grouped into a single entity, a 3x3 object called the stress tensor, $\mathbb{S}$,

$$\mathbb{S} = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}, \tag{64}$$

that can be added and multiplied with the same rules that apply to matrices. (Note this reuse of S to represent a stress, vs. a scalar tracer in Sec. 3.2 and 3.3., and the subscripts identify the element of the tensor, and not partial derivatives.) By convention, the first subscript on a stress component indicates the direction of the stress (the projection of the stress vector on to the unit vectors), and the second subscript indicates the orientation of the surface via the direction of the outward unit normal of the surface. [28]

A handy property of the stress tensor is that the matrix product $\mathbb{S}\cdot\mathbf{n}$ picks out the components of the stress vector acting on the face whose unit normal vector is $\mathbf{n}$. For example, let the unit normal for the upper face of the cube in Fig. (19) be $\mathbf{n_u}$, which thus has components (0; 0; 1) (the semicolon delimiters indicate that these elements are arranged in a column vector), and so $\mathbb{S}\cdot\mathbf{n_u} = (S_{xz};\ S_{yz};\ S_{zz})$, is a three element (column vector) of the components of the stress vector acting upon the upper face of the parcel. The force associated with this stress acting upon a differential area of the surface, da, is then

$$d\mathbf{F} = \mathbb{S}\cdot\mathbf{n}\, da, \tag{65}$$

[28] It can be argued that three pairs of the stress tensor components are equal, $S_{yx} = S_{xy}, S_{xz} = S_{zx}$ and $S_{zy} = S_{yz}$, at least for ordinary fluids such as air and water. This leaves six unique components and a stress tensor that is symmetric, $\mathbb{S} = \mathbb{S}'$. A little more on this below.

where the stress tensor is evaluated at the position of the differential area. The unit normal of the lower (downward looking) face of the control volume of Fig. (1) has components (0; 0; -1) and so the stress vector on that face has components $\mathbb{S}\cdot\mathbf{n_l} = -(S_{xz};\ S_{yz};\ S_{zz})$, where these components are evaluated at the position of the lower face. If it happened that the stress components were the same on the upper and lower faces, then the stress exerted by the overlying fluid on the upper face of the volume would be equal in direction and magnitude to the stress exerted by the fluid within the volume on the underlying fluid, in which case the net stress on the control volume, $\mathbb{S}\cdot\mathbf{n_u} + \mathbb{S}\cdot\mathbf{n_l} = \mathbb{S}\cdot(\mathbf{n_u}+\mathbf{n_l}) = 0$, would vanish.

Given Eq. (65) for the differential force, the net force on the fluid within the control volume may be computed by summing over the entire surface,

$$\mathbf{F} = \iint \mathbb{S}\cdot\mathbf{n}\, da, \tag{66}$$

where da is the differential area associated with the unit normal $\mathbf{n}$. Both $\mathbf{S}$ and $\mathbf{n}$ will, in general, vary over the surface. It is often desirable to express the net force as an integral over the volume of the object, rather than its surface, and this transformation is made by application of the tensor form of the divergence theorem,[29]

$$\mathbf{F} = \iint \mathbb{S}\cdot\mathbf{n}\, da = \iiint \nabla\cdot\mathbb{S}^{\mathbf{t}}\, dv. \tag{67}$$

Stress tensor in an ideal fluid, pressure only: The stress tensor has to be defined at every point in space and thus a tensor stress field is coincident with the velocity field. At first this seems a bit daunting; we had the goal of solving for the pressure and the velocity fields, and now it seems that we have to solve for a tensor field as well. However, as will see shortly, the stress tensor depends upon the velocity and the pressure in a straightforward way, so that nothing beyond the velocity and pressure fields will be required.

The stress vector due to pressure is just $-P\mathbf{n}$. Hence the stress tensor for a fluid that sustains only pressure forces, an ideal fluid, is simply

$$\mathbb{S}_{ideal} \;=\; \mathbb{P} \;=\; \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}, \tag{68}$$

an isotropic tensor (the same in all directions). The net pressure force is then

$$\boxed{\mathbf{F_{pressure}} \;=\; \iint \mathbb{P}\cdot\mathbf{n}\, da \;=\; \iiint \nabla\mathbb{P}\, dv \;=\; -\iiint \nabla P\, dv} \tag{69}$$

where the divergence theorem was applied to convert the surface integral into a volume integral and the

[29] This is probably not familiar, so I'll try to make it plausible. Each row of the stress tensor may be considered to be a vector, S_{1j}, S_{2j}, S_{3j}, so that there are three vectors whose components are numbered by the index j. Each of these vectors can be treated by way of the familiar divergence theorem for vectors, $\oint S_{ij} n_j da = \oint \frac{\partial S_{ij}}{\partial x_j} dv$ with i = 1, 2 or 3. Thus each row (or vector) of the tensor goes through separately. When we represent the divergence operator using symbolic notation, with ∇ a row vector operator that will be multiplied on the left of $\mathbf{S}$, then to operate on rows (each vector separately) we must operate on the transpose of $\mathbf{S}$, and thus $\mathbf{S^t}$ arises in Eq. (67). Later it will turn out that $\mathbf{S^t} = \mathbf{S}$.

last step holds only for isotropic tensors. The pressure force per unit volume is then

$$\mathbf{F}_{\mathbf{pressure}}/Vol \;=\; -\nabla P \;=\; -(\frac{\partial P}{\partial x}\mathbf{e_x} + \frac{\partial P}{\partial y}\mathbf{e_y} + \frac{\partial P}{\partial z}\mathbf{e_z}). \tag{70}$$

Stress tensor in a Newtonian, viscous fluid: In Sec. 6.1 it was claimed that the viscous shear stress in a simple shear flow in which the u component of velocity varied in the z-direction only (Figure 22) was $\tau_{xz} = \nu\frac{\partial u}{\partial z}$. There is nothing special about the z direction and if the u component of velocity varied in the y-direction, then there should arise a viscous stress in the x-direction that is exerted on the y-face of the parcel in exactly the same way. That viscous shear stress component would then be labelled

$$\tau_{xy} = \nu\frac{\partial u}{\partial y},$$

with a switch from S, indicating a stress generally, to τ to indicate a viscous stress in particular (you should make a sketch that shows this kind of shear flow and check whether the sense of the stress is given correctly by this equation).

An x-directed stress acting upon the x-face might be different, however, since this would involve a linear deformation rate rather than a shear deformation rate. For some fluids it is found experimentally that the viscosity for a linear deformation rate is not equal to the viscosity associated with shearing deformation, in much the same way that the bulk modulus of a solid is generally not equal to the shear modulus (Table 1, Sec. 6.1). While acknowledging that this is plausible for fluids as well, it will nevertheless be presumed here that there is only one viscosity, ν. Thus a linear deformation rate will produce or require a normal viscous stress,

$$\tau_{xx} = \nu\frac{\partial u}{\partial x}.$$

In most cases this normal viscous stress will be very much less than the normal stress associated with pressure.[30] With these three examples in hand, we can write the viscous stress tensor for Newtonian fluids in which the stress and rate of deformation are related linearly (Fig. 22),

$$\text{stress tensor for a Newtonian fluid:} \quad \boxed{\mathbb{T} \;=\; \nu\mathbb{G}} \tag{71}$$

[30] A viscous normal stress is somewhat harder to envision than is a viscous shear stress. But imagine a fluid column that is falling under the influence of gravity while restrained by a normal viscous (tensile) stress associated with the linear deformation rate of the elongating fluid column, e.g., the pitchdrop experiment (see footnote 38).

where the

$$\text{velocity gradient tensor:} \quad \boxed{\mathbb{G} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}} \tag{72}$$

The tensor $\mathbb{G}$, called the velocity gradient tensor in this context, is very useful in that it contains all of the first partial derivatives of the velocity field in a single, readily manipulated object. It will appear again on several occasions in Part 2.

To find the viscous force per unit volume, apply the tensor equivalent of the divergence theorem, Eq. (67),

$$\mathbf{F}_{\text{viscous}} = \iint \mathbb{T} \cdot \mathbf{n}\, da = \iiint \nabla \cdot \mathbb{T}\, dv. \tag{73}$$

The force per unit volume, written in vector and component form is then

$$\mathbf{F}_{\text{viscous}}/Vol = \nu \nabla^2 \mathbf{V} = \nu \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{pmatrix}. \tag{74}$$

(Note that ∇^2 operating on a vector yields another vector, whose components are the Laplacian of the components of the original vector.) In all of the above it has been assumed that the viscosity is a constant.

The expression (74) for the viscous force per unit volume may at first look formidable because of the second derivatives and the large number of terms involved. But notice that it is linear and does not couple the components together, i.e., in the x-component equation there appears only the x-component of velocity, albeit differentiated with respect to all three spatial coordinates. The viscous force per unit volume may be familiar to you as the diffusion term of the elementary heat diffusion equation, and indeed, the momentum components are diffused through a fluid in laminar flow just the way that thermal energy is diffused through a solid.[31] Diffusion, or viscosity in the case of fluid momentum, acts to smooth out lumps and bumps in the velocity profile (in any direction) with the rate given by the viscosity times the Laplacian.[32]

[31] It bears emphasis that this is valid for molecular diffusion in a laminar flow in which Eq. (130) applies. Turbulent motion may cause mixing in a fluid that is often parameterized as a diffusion process, Fig. (22), in which case the equivalent viscosity (stress/shear) is likely to be a complex function of the flow, and not just a physical property of the fluid.

[32] Even a spatially homogeneous but asymmetric stress tensor would a induce very large (infinite) angular acceleration locally

Navier-Stokes equations: Given this specification of pressure and viscous forces,

$$\mathbf{F} \;=\; \mathbf{F}_{pressure} + \mathbf{F}_{viscous} \;=\; -\nabla P \;+\; \nu\nabla^2\mathbf{V},$$

the Eulerian form of the momentum balance equation is then, including the body force due to gravity,

$$\text{Navier-Stokes equation:} \qquad \boxed{\frac{\partial \mathbf{V}}{\partial t} \;+\; (\mathbf{V}\cdot\nabla)\mathbf{V} \;=\; \frac{-1}{\rho}\nabla\mathbf{P} \;+\; \frac{\nu}{\rho}\nabla^2\mathbf{V} \;-\; g\mathbf{e_z}} \tag{75}$$

If the density is regarded as variable, then we can not write the velocity of the Euler fluid in flux form directly, but use instead the momentum density, i.e., $\rho\mathbf{V}$. By adding the mass conservation equation to the momentum equation and with minor rearrangement (that you should verify),

$$0 \;=\; \frac{\partial}{\partial t}\begin{pmatrix} \rho u \\ \rho v \\ \rho w \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \rho u^2 \;+\; p \;-\; \nu\dfrac{\partial u}{\partial x} \\ \rho uv \;-\; \nu\dfrac{\partial v}{\partial x} \\ \rho uw \;-\; \nu\dfrac{\partial w}{\partial x} \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} \rho vu \;-\; \nu\dfrac{\partial u}{\partial y} \\ \rho v^2 \;+\; p \;-\; \nu\dfrac{\partial v}{\partial y} \\ \rho vw \;-\; \nu\dfrac{\partial w}{\partial y} \end{pmatrix} + \frac{\partial}{\partial z}\begin{pmatrix} \rho wu \;-\; \nu\dfrac{\partial u}{\partial z} \\ \rho wv \;-\; \nu\dfrac{\partial v}{\partial z} \\ \rho w^2 \;+\; p \;-\; \nu\dfrac{\partial w}{\partial z} \;+\; \Phi \end{pmatrix} \tag{76}$$

where the gravitational potential $\Phi = \rho gz$. [33]

There is a lot of symmetry in the momentum equations, and they can be written in a much more compact (though abstruse) form by introducing a momentum flux tensors for the advection terms and gravity, added to the viscous stress and pressure stress tensors, $\mathbb{T}$ and $\mathbb{P}$ (Section 2.3). The momentum flux tensor

$$\mathbb{A} = \rho\begin{pmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{pmatrix},$$

can be written as the direct vector product

$$\mathbb{A} \;=\; \mathbf{V}\mathbf{V}^t \tag{77}$$

within a fluid, unless balanced by some other (exotic) external force that exerts a torque. The inference is that for non-exotic fluids such as air and water, the stress tensor must be symmetric. To insure this, $\mathbb{S}$ is often evaluated in a form that enforces symmetry, $\mathbb{S} = 1/2\nu(\mathbb{G}+\mathbb{G}')$ so that $\mathbb{S}_{ij} = \mathbb{S}_{ji}$. The first form shown here in Eq. (71) isn't wrong, but it may be impractical for calculations.

[33] Most arbitrary quantities are *not* subject to a strict conservation law. For example, in a two particle collision, the kinetic energy will be conserved only in the special case that the collision is 'elastic', so that no energy is lost to deformation, acoustic waves, etc. On the other hand, if the particles stick together after the collision, then the kinetic energy will decrease by an amount that depends upon the particular conditions of mass and initial velocity. Thus the kinetic energy and higher moments of the velocity are not conserved in most collisions or during mixing events in a fluid. More often a conservation law will not obtain because of the presence some external source, e.g., gravity, that does not vanish with the fluid velocity and hence the global integral need not be conserved. In that case we should probably call the governing equation the momentum 'balance' or 'budget' rather than 'conservation ', though this distinction is often ignored.

where $\mathbf{V}$ is a 3x1 column vector and the transpose $\mathbf{V}^t$ is a 1x3 row vector; hence $\mathbb{A}$ is 3x3. The gravitational tensor is just

$$\mathbb{Y} = \rho g z \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Define the total momentum flux tensor

$$\mathbb{M} = \mathbb{A} + \mathbb{P} - \mathbb{T} + \mathbb{Y}.$$

The tensor form of the Eulerian momentum balance is then

$$\text{momentum balance, conservation form:} \quad \boxed{\frac{\partial(\rho \mathbf{V})}{\partial t} + \nabla \cdot \mathbb{M} = 0} \tag{78}$$

where this ∇ is a three element row vector, $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. There is little gain in this for the pressure and gravitational terms. But having written the momentum balance equation in tensor form we now can assert two important properties: 1) The tensor equation (78) is valid in any (Cartesian) coordinate system, assuming that it is correct in one coordinate system, and 2) The components of the tensors transform under a rotation of the coordinate axes as

$$\mathbb{A}' = \mathbf{V}'(\mathbf{V}')^{\mathbf{t}} = \mathbb{R}\,\mathbb{A}\,\mathbb{R}^t, \tag{79}$$

where $\mathbb{R}$ is the rotation tensor.[34]

3.4.5 The flux form of the momentum equation; Reynolds number

Consider a very simple example; suppose that the aim is to account for the $x-$component of momentum within a control volume of $x-$length l, and width and height w and h, and suppose too that the y and z variations of the fluxes can be ignored (Fig. 12). The $x-$component of the total momentum flux is $\mathbb{M}_{(1,1)}$,

$$\mathbb{M}_{(1,1)} = \rho u^2 + p - \nu(\partial u/\partial x). \tag{80}$$

The integral momentum budget for this fixed control volume is then

$$\begin{aligned} lwh\frac{\partial \rho u}{\partial t} &= wh(\,\mathbb{M}_{(1,1)}(x=l) - \mathbb{M}_{(1,1)}(x=0)\,) \\ &= wh(-\rho u^2 - p + \nu\frac{\partial u}{\partial x})\,|_{x=l} - (-\rho u^2 - p + \nu\frac{\partial u}{\partial x})\,|_{x=0}). \end{aligned} \tag{81}$$

[34] We can readily infer this last result from the form of the momentum flux tensor, Eq. (77). If the velocity vector (components) transform under rotation of coordinate axes as $\mathbf{V}' = R\mathbf{V}$, and given that $(\mathbf{V}')^t = (R\mathbf{V})^t = \mathbf{V}^{\mathbf{t}}R^t$, then by substitution into the middle term of Eq. (79) and using the associative property of matrix multiplication we verify the transformation rule for Cartesian tensors. (A good review of tensor algebra is Ch. 2 of Kundu and Cohen (2001) noted in footnote 10.)

Dividing through by the volume, lwh, taking the limit that l becomes small, application of mass conservation (Eq. (60) and *voila*, out pops the differential form of the momentum balance,

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{\rho}\frac{\partial^2 u}{\partial x^2}. \tag{82}$$

The only slightly disagreeable part of this procedure is that the forcing terms were applied to points, a control volume, rather than to a material fluid volumes, but the RTT leads to the same thing.

The value of this derivation is that it may be most natural to make assertions about the physical properties of a flow at the first stage of this procedure, Eq. (80), where the flux is prescribed. Here we have included all three of the terms that might contribute to momentum flux, but often this is not necessary or appropriate. In most flows the pressure gradient is of leading importance, and the interesting question is likely to be how the advective and diffusive terms compare to one another. To make a rough estimate of these terms requires an estimate of the length scale, L, over which the current component varies by about 100%. That is, what value of L is appropriate to the relations,

$$\frac{\partial u}{\partial x} \text{ is } O(\frac{U}{L}) \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} \text{ is } O(\frac{U}{L^2}),$$

where the big O notation means the order of magnitude without regard to the sign. For example, if $u = A\,sin(kx)$, then $\partial u/\partial x = A\,k\,cos(kx)$ is $O(Ak)$ and so $U = A$ and $L = k^{-1} = \lambda/2\pi$. If the aim is a rough estimate, we might ignore the factor 2π and just say that L is $O(\lambda)$. Of course, when factors of 2π or 2 cascade, as they do for the second derivative, then this could eventually lead to trouble. But with that in mind, let's proceed to estimate the ratio of the advective term to the diffusive term as

$$\frac{\text{advection}}{\text{viscous drag}} = \frac{u\frac{\partial u}{\partial x}}{\nu\frac{\partial^2 u}{\partial x^2}} \text{ is } O(\frac{u^2/L}{\nu u/L^2}) = O(\frac{UL}{\nu}) = O(Re), \tag{83}$$

where Re is the important nondimensional,

Reynolds number: $$\boxed{Re = \frac{\text{advection}}{\text{viscous drag}} = \frac{UL}{\nu}}$$

The Reynolds number is liable to be important whenever viscosity is at issue. The most uncertain aspect of Re will often be the length scale, L. For the wave-like motion shown in the cover graphic, we can estimate that $U \approx 0.1$ m s^{-1}, and that $L \approx 100$ km, roughly. Using the known kinematic viscosity of water, $\nu = 10^{-4}$ m^2 s^{-1}, then $Re \approx 10^8$. No matter how crude the estimate of L, it is unmistakable that the viscous drag term is much, much, much smaller than the advection term in this instance. After making a few such estimates, you will be entirely justified in dropping (without even mentioning) the viscous, diffusive contribution to the momentum flux in the case that the momentum budget is evaluated for such large scale motions.

You might well wonder if the viscosity term is ever important. Yes, but the viscous term is of leading importance only for scales of motion that are very, very small compared to the basin scale of oceanic or

atmospheric motions. Recall the flow in a tea cup; the smallest scales of motion are vortices having $L = 0.005$ m, roughly, and a typical speed $U = 0.05$ m s^{-1}, and hence $Re \approx 1/2$. You may have noticed that these small scales of motion are damped rather quickly, within a few seconds. Can this be attributed to viscosity? The time scale required for diffusion to propagate a signal over a distance L is roughly $t_{diff} = L^2/\nu$, deduced from dimensional analysis or from a solution of the diffusion equation, while the rotation time for a vortical motion is $t_{adv} = L/U$. The ratio of these time scales is $t_{diff}/t_{adv} = Re$, once again the Reynolds number, and so yes, viscous diffusion of momentum is evidently sufficient to damp the smallest vortical motions in a few seconds, or the order of several vortex rotations. What we can go on to infer from this, and verify by observation, is that larger vortices last longer, all else equal.

3.4.6 Energy conservation; the First Law of Thermodynamics applied to a fluid

As a final application we will consider the balance of thermal energy in a compressible fluid. The First Law of Thermodynamics keeps track of several kinds of energy storage and energy exchange that may occur between a given fluid volume and the environment, schematically,

$$\frac{dE}{dt} \;=\; WorkRate \;+\; HeatSource \tag{84}$$

where $E \;=\; \rho C_v T$, is the internal energy, and C_v is the heat capacity of the fluid in an isovolume process, taken to be a known physical property of the fluid, and a constant for this purpose. *WorkRate* is the rate at which mechanical work is done by the environment on the fluid volume. In the case of an Euler fluid, this can be due only to the rate of work by pressure, $-P\mathbf{n}\cdot\mathbf{V}$, evaluated over the surface of the volume. *HeatSource* is the divergence of heat fluxes due to radiation, $-\nabla\cdot\mathbf{Q}$, and conduction, $-\kappa\nabla T$, where κ is the thermal conductivity of the fluid, and to a body source due to the dissipation of kinetic energy to internal energy, ε. In most cases ε is negligibly small compared to the heat fluxes or *WorkRate* and will be omitted from here on. Our final result will be appropriate for geophysical flows but not completely general. The approximate balance of thermal energy is then:

$$\frac{DE}{Dt} \;=\; C_v \frac{D}{Dt} \iiint\limits_{material} \rho T \; dVol \;=\; -\oint P\mathbf{n}\cdot\mathbf{V}\; da \;-\; \oint \kappa\nabla T\cdot\mathbf{n}\; da \;+\; \iiint\limits_{material} -\nabla\cdot\mathbf{Q}\; dVol. \tag{85}$$

Applying the divergence theorem to the surface integrals and the RTT and then collecting terms under the volume integral yields the differential form of the thermal energy balance in field coordinates:

$$\rho C_v \frac{DT}{Dt} \;=\; -P\nabla\cdot\mathbf{V} \;+\; \kappa\nabla^2 T \;-\; \nabla\cdot\mathbf{Q}. \tag{86}$$

To simplify this a little further, the fluid will be presumed to be an ideal gas described the equation of state $P = \rho RT$, where $R = C_p - C_v$ is the universal gas constant and C_p is the heat capacity at constant pressure (Table 1). The divergence of fluid velocity may be eliminated by use of the mass conservation equation (43) and then the equation of state,

$$-P\nabla\cdot\mathbf{V} \;=\; -\frac{P}{\rho}\frac{D\rho}{Dt} \;=\; \frac{P}{\rho}\frac{\partial\rho}{\partial T}\Big|_P \frac{DT}{Dt} \;=\; -\frac{P}{\rho}\frac{P}{RT^2}\frac{DT}{Dt} \;=\; -\rho R\frac{DT}{Dt}.$$

When this is substituted into Eq. (86) the result is the

$$\text{first law of thermodynamics for an ideal gas:} \quad \boxed{\rho C_p \frac{DT}{Dt} = \kappa \nabla^2 T - \nabla \cdot \mathbf{Q}} \tag{87}$$

where notice the heat capacity is now C_p in place of C_v, which is significant for a gas. For many liquids and solids the distinction between C_p and C_v is not large.

3.4.7 Vorticity balance

The curl of the three-dimensional velocity gives the

$$\text{three-dimensional vorticity:} \quad \boxed{\xi = \nabla \times \mathbf{V}} \tag{88}$$

The curl of the momentum equation gives the corresponding

$$\text{three-dimensional vorticity balance:} \quad \boxed{\frac{D\xi}{Dt} = (\xi \cdot \nabla)\mathbf{V} - (\nabla \cdot \mathbf{V})\xi + \frac{\nabla\rho \times \nabla P}{\rho^2} + \nu \nabla^2 \xi} \tag{89}$$

This requires rearranging terms only, with no new approximations or additions, and hence this is not a new equation independent of the momentum balance. It takes some experience to appreciate that vorticity balance can be a very useful guide to the evolution of an important subset of the flows governed by the momentum balance. Vorticity is present in some but not all of the phenomonenon contained within the former. Specifically, the curl operator gives zero when operating on any vector that is derivable from a scalar potential, and most importantly the pressure gradient, ∇P, and thus

$$\nabla \times \nabla P = 0.$$

The curl operator thus eliminates the often very signifcant effects of a pressure gradient (though with an important exception noted below) and also of gravity. Fluid flows that are due mainly to these two important forces, e.g., gravity waves, are thus invisible to (filtered out of) the vorticity equation, while most ocean currents and winds (and many, many other phenomenon) go through unscathed.

The terms of the vorticity equation are

$(\xi \cdot \nabla)\mathbf{V}$, which represents the tipping (or turning) and stretching of the vorticity vector that arises when there is velocity shear in the direction of the vorticity vector. This is a mechanism for turning horizontal vorticity into the vertical (as occurs in tornadogenesis) and important for distributing vorticity within small-scale, three-dimensional flows. Not considered further here.

$-(\nabla \cdot \mathbf{V})\xi$, which accounts for the amplifiction of vorticity when the fluid is stretched or compressed. This is very important in the vorticity balance of the oceanic flow of Fig. 1, and discussed further in Sec. 4.

$(\nabla\rho \times \nabla P)/\rho^2$, is the vorticity expression for flows that are generated by density differences, e.g., a

diurnal sea breeze circulation. This is sometimes caled a solenoial term, after the crossing isolines of density and pressure that are present when this term is nonzero. Not pursued here.

$\nu\nabla^2\xi$ is the diffusion of vorticity that follows directly from the assumption of Newtonian diffusion of momentum. To this approximation, vorticity diffuses through a fluid just like heat (temperature differences). In Part 2 we will see a model of vorticity diffusion that accounts for the evolution of the irrotational vortices generated in the tea cup experiment.

Notice that the first, second, and fourth terms are proportional to vorticity. They represent processes that can change vorticity, but can not generate vorticity from an irrotational flow (boundary conditions aside). That is not the case with the solenoidal term.

3.5 Remarks on the Eulerian equations

The momentum equations (75) have two terms that are characteristic of fluid flows, the pressure gradient term and the advection terms. Pressure is a scalar, and the gradient of the pressure appears in each of the component equations. A localized pressure perturbation in a three-dimensional flow is thus likely to induce motion in all three components. The pressure gradient thus acts to couple the component equations, and is generally the physical process that allows for (or enforces) mass conservation; as fluid converges into a given volume the pressure will rise and so produce a compensating pressure-driven divergence, e.g., acoustic and gravity waves if high or low frequency. The pressure gradient term is linear, and does not, in and of itself, present any special mathematical difficulty.

Advection endows many fluid flows with rich spatial structure and complexity, and advection is the process that most distinguishes a fluid flow from solid mechanics. The advection terms are nonlinear, in general, and hence the advective terms stymie most of the familiar PDE solution techniques that require superposition of solutions. There are many cases in which the advection terms are demonstrably much, much smaller than the pressure gradient term, and so they may be omitted to leave a linear momentum balance equation. A linearized system of that sort omits some of the distinctive character of fluid flow but may nevertheless be entirely appropriate for the analysis of small amplitude, wave-like motions, especially.

The (Eulerian) momentum, mass and energy equations are impressive, and it might seem that writing them out in full would be a significant step towards the solution of a fluid mechanics problem. Well, yes and no. An understanding of the origin and the meaning of these equations is a vital first step toward understanding fluid mechanics. But it is essential to appreciate that these equations are extremely general, and their content is as their name implies — that momentum, mass and energy are conserved consistent with classical physics. And here's the punch line, there is *nothing more implied than that*. Momentum conservation may occur by virtue of wave motions or swirling vortices or seemingly random turbulence. It is not unusual that a flow (even in a tea cup) may include all three phenomena at once.

3.6 Key ideas

1) The vast majority of theory and modelling of fluid mechanics is carried out within an Eulerian framework.

2) The independent variables of an Eulerian model are fixed spatial coordinates (field coordinates) and time. The dependent variables are the fluid velocity, pressure, density, etc. defined on the fixed coordinates.

3) The central elements of an Eulerian model or theory include the Eulerian velocity, the material time rate of change operator, and the conservation laws. These are derivable from a Lagrangian origin via the FPK, the material time derivative holding parcel identity fixed, and the Reynolds Transport Theorem.

4) The material time derivative includes an advection term, the dot product of the fluid velocity and a tracer gradient, that represents the effects of fluid flow in an Eulerian conservation equation. In general, the velocity and the tracer gradient are unknowns, which makes the advection term nonlinear (more precisely, bilinear).

5) The advection term may be written in flux form by combining with the mass conservation relation. This helps make clear that for most purposes it is the divergence of the flux that is the relevant quantity, rather than the flux itself.

6) An Eulerian solution for density, say, comes in one complete piece, $\rho(x,t)$, that includes the effects of stretching and advection (ideal fluid). This may be compared with the Lagrangian solution that comes in two pieces, the density itself which includes the effects of stretching, and the trajectory.

7) The normal and tangential stresses on a fluid parcel in R^3 may be written as a 3x3 tensor. The diagonal elements include an isotropic pressure, and the off-diagonal elements are tangential stresses, e.g., viscous stress.

8) Many fluids, including air and water, are Newtonian in the sense that viscous stress in laminar flows is found to be approximately proportional to the velocity shear times the fluid viscosity, a thermodynamic property of the fluid.

9) Turbulent flow is far more common than is laminar flow. Under controlled circumstances (e.g., unperturbed, smooth boundary) the transition from laminar to turbulent flow for a given geometry may occur at a critical Reynolds number. In a turbulent flow, the stress depends upon flow properties other than just the local shear.

3.7 Problems

1) Here's another simple Lagrangian solution $\tilde{x} = a(e^t + 1)$ with a a constant. Compute and interpret the Lagrangian velocity $\tilde{(u)}(a,t)$ and the Eulerian velocity field $u(x,t)$. Suppose that two parcels have initial positions $a = 2a$ and $2a(1+\delta)$ with $\delta \ll 1$; how will the distance between these parcels change with

time? How is the rate of change of this distance related to $u(x,t)$? (Hint: consider the divergence of the velocity field, $\partial u/\partial x$.) Suppose the trajectories are instead $\tilde{x} = a(e^t - 1)$.

2) A couple of questions for you. (1) Can you show that the material derivative of parcel position is just the velocity? In this respect, the material derivative could also be said to be more fundamental than is the FPK. (2) Suppose a one-dimensional domain as above, and a fluid variable that has an initial, Lagrangian distribution $b = ca$, with trajectories $\tilde{x} = a + Vt$, and V a constant. Given that b is conserved in every material volume, find the equivalent material derivative in the Eulerian system. Interpret the local rate of change of b. (3) Suppose instead that $Db/Dt = a$, with a a constant; how does $b(x,t)$ evolve?

3) The Eulerian conservation equation for mass in an ideal fluid flow includes an advection term and a divergence term, Eqs. (44). Assume that the flow includes a spatially-uniform component, U_o. Compare the effects of this U_o within the advection process, (50) and the divergence process (51). The solutions of Fig. (8) were evaluated assuming $U_o = 0$. How would the solutions change if instead $U_o = 0.2$, say.

4) You should fill in the steps leading up to Eq. (59) to verify that this is true whenever the extensive property, say H, is the volume integral of ρh, where ρ is mass density and h is the corresponding intensive property.

5) After deriving the vector form of the pressure gradient force per unit volume we went back and derived the z-component of the pressure force without the use of vector or tensor notation. You should do the same for the viscous force per unit volume in the case shown in Fig. 22, a flow in the x-direction only, and with shear in the z-direction only. Find the viscous force on the upper and lower faces of a parcel, then the sum, and allow the dimensions to shrink to infinitesimal. Locate the resulting term in the full 3-dimensional equation, (74). Suppose that the flow has reached a steady state, as in Fig. 22. What is the profile of viscous stress throughout the fluid? Now imagine that the stress has just been imposed at the surface and the flow is developing. Qualitatively, what is the stress profile in this transient case? The lower boundary condition in Fig. (22) is presumed to be no-slip, so that $u(z = 0) = 0$, which is appropriate for real fluids having a finite viscosity. What is the equivalent boundary condition for a heat diffusion problem? What are the corresponding, plausible surface boundary conditions?

4 An advection problem, two ways

This section will develop the Lagrangian and Eulerian perspectives on an elementary advection problem. To keep the focus on the concepts and methods, this problem will be kept simple: a passive tracer, c, in a prescribed, ideal fluid flow in R^1, with space and time field coordinates (x,t). There will be no diffusion, and to start, no source for the tracer, and so the tracer dynamics will be termed adiabatic. There will be no requirement for mass or volume conservation, and no pressure gradient, and hence the fluid parcels of this model do not interact with one another as they do in a real fluid flow. With these restrictive assumptions in place, the governing equation for the tracer c is

$$\frac{Dc}{Dt} = 0, \tag{90}$$

where D/Dt is the material derivative, and could be evaluated in Lagrangian or Eulerian coordinates. For the first version of this problem, take the current to be

$$U = constant, \tag{91}$$

a steady, uniform translation toward positive x. The initial condition data (IC) will be given at $t = 0$ along the x axis,

$$c(x, t = 0) \;=\; \exp(-x^2/2), \tag{92}$$

a Gaussian hump with amplitude and width of 1. The set of three equations (90) - (92) define an advection/initial value problem for the tracer c in which advection is the only active process.

4.1 The Lagrangian solution

The Lagrangian version of this problem is very straightforward. The tracer in material coordinates is $\tilde{c}(a,t)$, with a the initial position, and Eqs. (90) and (92) are

$$\frac{D\tilde{c}(a,t)}{Dt} \;=\; \frac{\partial \tilde{c}(a,t)}{\partial t} \;=\; 0, \quad \text{with an IC} \quad \tilde{c}(a, t = 0) \;=\; \exp(-a^2/2). \tag{93}$$

The governing equation states that $\tilde{c}$ does not depend upon time, and so

$$\tilde{c}(a,t) \;=\; \tilde{c}(a),$$

and given the IC (93), the Lagrangian solution for tracer concentration is just

$$\tilde{c}(a) \;=\; \exp(-a^2/2). \tag{94}$$

Each parcel (each a) has a tracer value assigned at the beginning, $t = 0$, and then keeps that value for all time, no matter how fast or how far it travels.

To make any use of this requires knowing also the parcel positions. Because the velocity U is spatially uniform, U may be treated as a Lagrangian velocity, and so from (91),

$$\frac{\partial \tilde{x}(a,t)}{\partial t} \;= U \quad \text{with IC} \quad \tilde{x}(a, t = 0) \;=\; a,$$

and hence

$$\tilde{x}(a,t) \;=\; a \;+\; Ut. \tag{95}$$

Eqs. (94) and (95) define the tracer concentration and the parcel positions, and thus make a complete Lagrangian solution. This solution may be readily interpreted in field coordinates, (Fig. 14) when the tracer concentration is envisioned to be a third (vertical) coordinate. The solution (Fig. 14) shows that the Gaussian hump in tracer concentration is translated to the right at the steady speed U, and without change in amplitude or width.

Toward the end of Sec. 2.4, a (soft) conclusion was that the Lagrangian momentum equations are

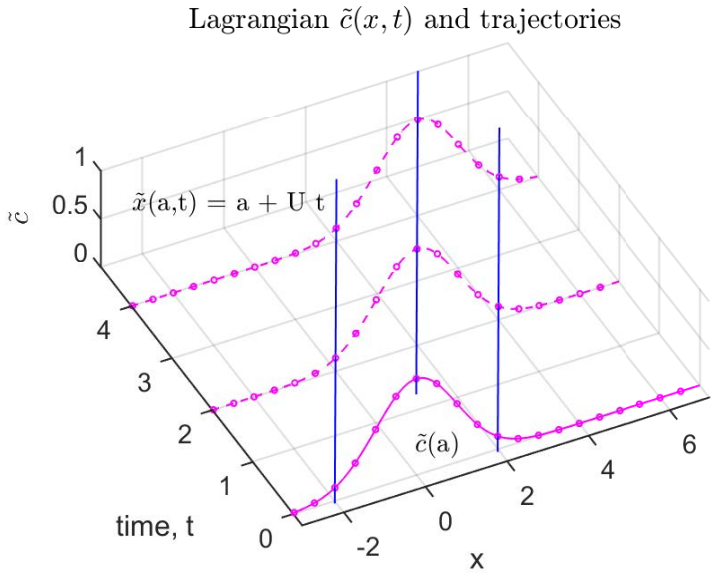

Figure 14: An Eulerian (field coordinate) interpretation of the Lagrangian solution Eqs. (94) and (95) of a tracer, $\tilde{c}(x,t)$, in a uniform flow $U = 1$ that translates fluid parcels toward positive x. The magenta dots are parcels at discrete values of a. The vertical coordinate represents the tracer concentration, which has no geometric significance. The blue lines are parcel trajectories, $\tilde{x}(a,t)$, evaluated at several discrete a, and which have a uniform slope $dt/dx = 1/U$. According to the governing equation, (93), the tracer value on parcels, $\tilde{c}(a)$, is constant in time as parcels move along their respective trajectories, $\tilde{x}(a,t)$.

often difficult to apply. This Lagrangian tracer solution might seem to contradict that conclusion, but notice that the current was here prescribed, and not computed from a momentum equation that would usually include a pressure gradient and diffusion. In this somewhat exceptional circumstance, the Lagrangian system is advantageous.

4.2 The Eulerian solution via the method of characteristics

In an Eulerian system, the tracer distribution is defined on a field, $c(x,t)$, with x and t the usual space and time coordinates. The governing equation corresponding to (90) is then

$$\frac{D\,c(x,t)}{Dt} \;=\; \frac{\partial c}{\partial t} + U\frac{\partial c}{\partial x} \;=\; 0. \tag{96}$$

This PDE is linear in the first partial derivatives (there is no occurrence of nonlinear terms such as $(\partial c/\partial x)^2$, or of $(\partial c/\partial x)(\partial c/\partial y)$), it has constant coefficients (the uniform velocity, U), and it is homogeneous (each term contains c or a derivative of c). An appropriate initial condition is

$$c(x,\, t=0) \;=\; exp(-x^2/2). \tag{97}$$

The solution of a first order PDE like (96) can be sought via the method of characteristics (MoC). The MoC seeks an Eulerian solution by taking account of the (in some ways) simpler tracer balance of a Lagrangian perspective. The concepts developed within the MoC are thus highly germane to the present discussion of Lagrangian/Eulerian methods, and they are frequently referenced in many fields of applied mathematics.

The basis of the MoC is a geometric interpretation of first order PDEs like (96). A generalized gradient

vector of the field $c(x,t)$ is

$$\nabla c \;=\; (\frac{\partial c}{\partial x},\; \frac{\partial c}{\partial t}). \tag{98}$$

The directional derivative of $c(x,t)$ in the direction of a vector $\Lambda \;=\; (\lambda_1,\; \lambda_2)$ (usually normalized) is just the vector product of Λ and this ∇c,

$$\Lambda \cdot \nabla c \;=\; \lambda_1 \frac{\partial c}{\partial x} \;+\; \lambda_2 \frac{\partial c}{\partial t}. \tag{99}$$

If it is given that

$$\Lambda \cdot \nabla c \;=\; 0, \tag{100}$$

then it may be inferred that Λ is perpendicular to the gradient of c, or just as well, that Λ is parallel to curves of constant c. Comparing (99) with (96) and identifying the components of the vector Λ with the coefficients of the PDE, in this case $\lambda_1 \;\propto\; U$ and $\lambda_2 \;\propto\; 1$, it is evident that c is constant along curves that are parallel to the vector field $(U,1)$. This is a key result, and the basis of the MoC. These curves are said to be the 'characteristic' curves of the PDE (96). In the case of an advection equation like (96), the coefficients are the velocity components, and the characteristic curves are parcel trajectories. However, the notion of characteristic curves is not dependent upon this interpretation, and the MoC applies to many first order PDE.

The MoC can be envisaged in three steps.

Step 1, calculate the family of characteristics curves that are parallel to the coefficient vector, here just $(U,\; 1)$. One way to do this, though it may seem a little indirect on first sight, is to seek a parametric form of the characteristic curves, $(x_\gamma(s),\; t_\gamma(s))$, where s is the parameter that will trace out the curve. (If the notion of a parameterized curve is unfamiliar, then you should take a quick detour to Problem 1) that starts with a refresher.) The important thing to note is the relationship between $x_\gamma(s)$ and $y_\gamma(s)$, and that s is the common link between them. The tangent vector to a parameterized curve is just

$$\Gamma \;=\; (\frac{dx_\gamma}{ds},\; \frac{dt_\gamma}{ds}),$$

and the condition that makes this tangent vector parallel to the coefficient vector $(U,\; 1)$ is that

$$\frac{dx_\gamma}{ds} \;=\; U, \quad \text{and,} \quad \frac{dt_\gamma}{ds} \;=\; 1. \tag{101}$$

The two ordinary differential equations (101) thus serve to define the family of characteristic curves of Eq. (96). Appropriate ICs are

$$x_\gamma \;=\; x_o \quad \text{when} \;\; s \;=\; 0, \;\; \text{and} \;\; t_\gamma \;=\; 0 \quad \text{when} \;\; s \;=\; 0. \tag{102}$$

These ordinary differential equations (ODEs) can be readily integrated to yield

$$x_\gamma \;=\; x_o \;+\; Us, \quad \text{and,} \quad t_\gamma \;=\; s. \tag{103}$$

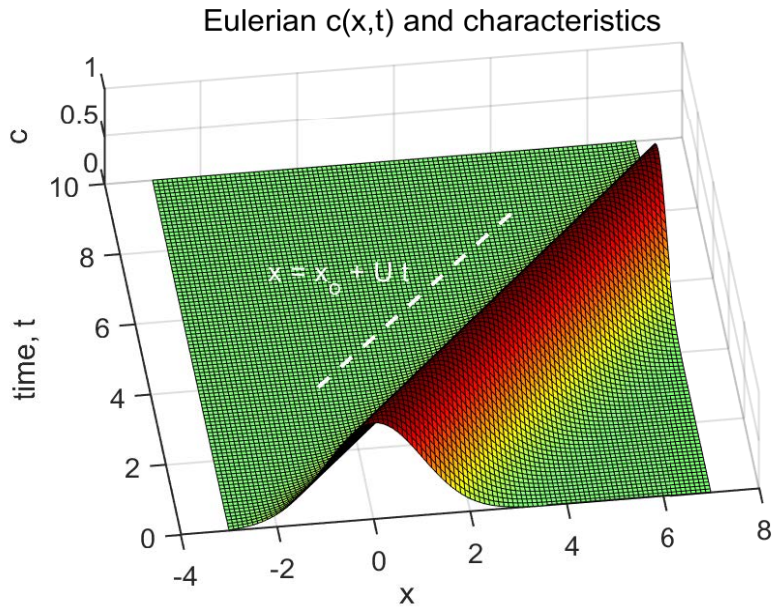

Figure 15: The Eulerian representation of a tracer field, $c(x,t)$, that is embedded in a uniform flow $U = 1$ that translates c to the right. The dashed white line $x = x_o + Ut$ is one characteristic curve of the flow; all other characteristics (different x_o) are parallel. Notice that $c(x,t)$ is constant in the direction parallel to a characteristic curve.

Eliminating s from (103) gives

$$x_\gamma = x_o + Ut_\gamma. \tag{104}$$

The family of characteristic curves along which $c =$ constant in this problem is given parametrically by Eqs. (103), or explicitly by Eq. (104). In the special case of a uniform velocity, U, considered here, the characteristics curves are straight lines having slope $dt_\gamma/dx_\gamma = 1/U$ (the slope is $\propto 1/U$ because the ordinate is time rather than x). The characteristic curves will not be parallel in the much more common circumstance that the current varies in space, Sec. 4.4. Notice that these characteristics are the same as the trajectories of the fluid parcels in the Lagrangian solution, cf. Fig. (15). The MoC follows from the expectation that the tracer balance will be simplest when sought along these characteristic curves. Indeed, in this adiabatic problem, the tracer values are simply advected (propagated or transported) along the characteristics without change.

Step 2, apply the boundary/initial data at some point along each of the characteristic curves: if we know the tracer at any point on a characteristic, then we know it at all points on that characteristic. Here that information is given by initial data, Eq. (97), that specifies $c(x,\ t = 0)$. Eliminate x_o from (92) using Eqs. (104) to find the explicit solution,

$$c(x,t) \ = \ \exp\left(-(x-Ut)^2/2\right). \tag{105}$$

Since the tracer balance was presumed to be adiabatic, the Gaussian distribution of $c(x,t)$ that was prescribed in the IC is shifted steadily to the right at the rate U, and with no change in amplitude or width of the initial hump, Fig. (15). This is equivalent to the Lagrangian solution of Sec. 4.1, as it should be.

Step 3 is to account for a source, postponed to a later example in Sec. 4.4.1.

4.3 Comments on these Lagrangian and Eulerian solutions

As emphasized on several occasions, Lagrangian and Eulerian solutions are quite different conceptually, even when they describe the same 'flow'. The Eulerian solution for the tracer is complete in one piece, Eq. (105). The Eulerian solution contains within it the effects of the fluid motion, the advection process, which is all that happens in the present problem. Compare this with the Lagrangian solution that comes in two pieces: the tracer value on moving parcels Eq. (94), and the position of those parcels, Eq. (95). The latter is the fluid motion. This two-part Lagrangian approach simplifies the tracer equation, but, you have to know the trajectories.

As we worked through the two solutions above, the intent was to be consistent within the Lagrangian system while in Sec. 4.1 — all about parcels with no fields — or the Eulerian system in Sec. 4.2 — all about fields but no parcels. When it is time to interpret or display a solution, there is no bar to mixing concepts; we can say c is constant along characteristics in an ideal fluid flow, or, equivalently, parcels conserve their tracer value as they move along trajectories in adiabatic flow. And, we can usually interpret a Lagrangian solution in field coordinates, as in Fig. (14). However, it is good practice to be conscious of mixing Lagrangian and Eulerian concepts when working toward a solution.

If you have been comparing these two solutions methods, then you are probably thinking that the MoC is somewhat abstract (formal and mathematical as opposed to physical reasoning), and more trouble than it was worth. In the present elementary problem, probably so. However there are some important phenomena that are best treated or interpreted via the MoC, an example being shock waves in Sec. 4.4.2. And, the notion of characteristics comes up fairly often in applied mathematics. You may also be harboring some doubt that the elementary advection problem examined to here could have anything to do with a genuine fluid mechanics problem; there was no reference to mass and momentum conservation, there was no accounting for a pressure gradient, and the fluid velocity was simply prescribed. True again, but there is more practical value in this very idealized problem than may be apparent on first sight. In every numerical integration that solves complex fluid systems, there comes the crucial task of advecting the fluid property fields, momentum, density, tracers, etc., with a velocity field that will be known. That very important piece of the numerical integration process — advection *per se* — is what we have considered in the problems above, and the ideas developed here will be put to good, practical use in Part 2, Sec. 4.

4.4 More advanced problems via the method of characteristics

The MoC can give useful insight into a variety of interesting phenomenon, including flows in higher dimensions and in cases with external forcing. The method is applicable to model equations that are first order and linear in the partial derivatives, e.g.,

$$\frac{\partial c}{\partial t} + u(c,x,y)\frac{\partial c}{\partial x} + v(c,x,y)\frac{\partial c}{\partial y} = Q(c,x,y), \tag{106}$$

but that may contain non-constant and nonlinear coefficients as well as a source or sink, Q.[35] Notice that the coefficients of the first order terms in (106) are here said to be u and v, as if velocity components in an advection equation, however that is not necessary. The governing equation (106) can be reduced to a set of ODEs beginning with the characteristics, which are just as seen in Sec. 4.2, one per independent variable,

$$\frac{dt_\gamma}{ds} = 1, \qquad \frac{dx_\gamma}{ds} = u(c,x,y), \quad \text{and} \quad \frac{dy_\gamma}{ds} = v(c,x,y), \tag{107}$$

and when a source is present, one additional ODE

$$\frac{dc}{ds} = Q(c,x,y), \tag{108}$$

that is discussed next. The solution of the characteristic equations (107) may not be trivial as they were in Sec. 4.2 where the current was uniform.[36]

4.4.1 With a source

To see the effect of a source (or if negative, a sink), Q, consider the one-dimensional version of Eq. (106) with IC as in Sec. 4.2, and now with an external source that causes a linear decay, say

$$\frac{Dc}{Dt} = Q = -\alpha c \tag{109}$$

where α is a positive constant. This non-adiabatic model could represent heat transfer from the fluid to the environment when the temperature difference is proportional to c, i.e., c is a temperature anomaly with respect to an infinite sink. In that case, c should relax back to the environmental temperature at long times, $t >> \alpha^{-1}$.

Step 3 is to evaluate the effect of the source. Consider that x and t are functions of s and so $c(x(s),t(s)) = c(s)$ is a composite function of the inner independent variable, s. The total derivative of c with respect to s is, by the chain rule,

$$\frac{dc}{ds} = \frac{\partial c}{\partial x}\frac{dx}{ds} + \frac{\partial c}{\partial t}\frac{dt}{ds}. \tag{110}$$

[35] There is no pressure gradient! As a consequence, fluid parcels can pile up, one on top of another, with no consequences (although a shock may form, Sec. 4.4.3). This is very different from the dynamics of what in Sec. 1.1 was dubbed 'ordinary fluids', water and air, which would develop a localized, high pressure anomaly that would sooner or later inhibit the further pile up of fluid. How fast this pressure anomaly develops and how fast the flow responds determines the speed of gravity waves and acoustic waves supported by the fluid. Is there any real fluid system that might be treatable by the very reduced dynamics (essentially advection only) considered in this discusion of the method of characteristics? The very high speed flow of a dilute gas as occurs in a rocket nozzle.

[36] The limitation to first order PDEs may seem severe, but first order equations are very common and important in physics and engineering, well beyond the (extensive) realm of fluid mechanics. Many kinds of second order wave equations can be factored into a pair of first order PDEs and then treated by the MoC (but not here). Where the MoC comes up wanting is cases when the governing equation has both first and second order partial differentials, as occurs when diffusion is acknowledged. Such parabolic systems do not entail a well-defined propagation speed, rather the entire domain is affected instantaneously by the diffusion process, and so not described well by characteristics.

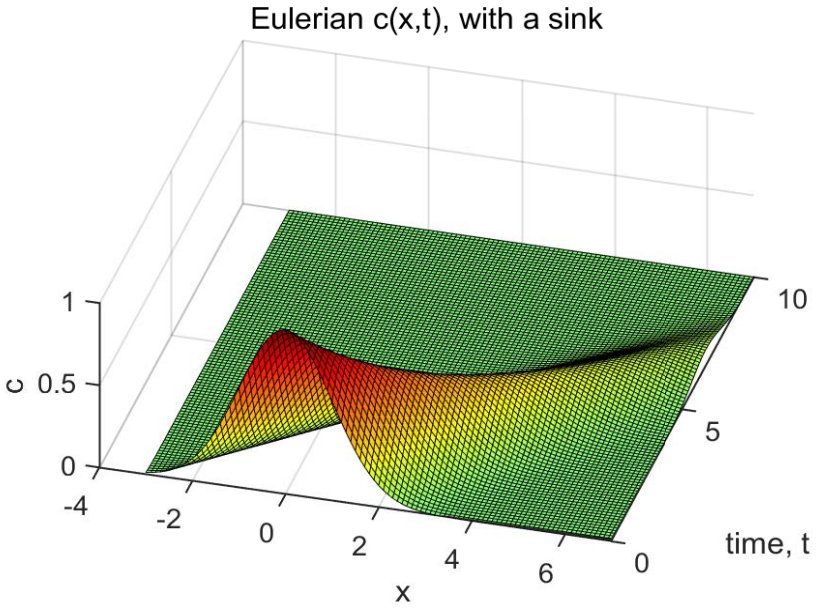

Figure 16: The Eulerian representation of a tracer field, $c(x,t)$, in a uniform flow $U = 1$ that has characteristics that are straight lines with slope $1/U$ as in the previous figure. There is a sink $Q = -\alpha c$ that causes an exponential decay of the tracer along the charactersitics curves. Here the coefficient $\alpha = 1/3$.

If $dx/ds = U$, and $dt/ds = 1$, then (110) is the directional derivative of c in the direction of the characteristic curves. In the previous, adaiabtic problem (Sec. 4.2) the tracer was constant along characteristic cuves. In this case, c is not constant, but instead decreases because of the sink, Q. Rewriting (110) using (109) gives

$$\frac{dc}{ds} = -\alpha c, \tag{111}$$

for which a suitable IC is $c(s = 0) = \exp(-x_o^2/2)$. This may be readily integrated to find c along the characteristics,

$$c(s) = \exp(-\alpha s)\exp(-x_o^2/2).$$

The characteristics are unchanged, since c is a passive tracer that has no effect on the velocity. Combining this with the previous, adiabatic solution (105), noting that $s = t$, and eliminating x_o yields

$$c(x,t) = \exp(-\alpha t)\exp(-(x-Ut)^2/2).$$

Thus the tracer concentration decreases exponentially in time along the characteristics on account of the sink (Fig. 16).

The MoC made this solution process seem easy since it broke up the PDE into two ODEs, one describing the effects of advection, which propagates c along characteristic curves without change, and the second describing the effects of the sink, Q, which in this case causes an exponential decrease in time along a characteristic. It is no accident that this looks and feels a lot like the Lagrangian perspective and solution of Sec. 4.1, however, the MoC does not rely upon the transport process being the motion of fluid parcels.

4.4.2 Variable coefficient and non-conservation

The two problems considered above could have been solved without invoking the MoC since they have a clear and simple physical interpretation. Now consider an equally simple problem where a physical interpretation is not so obvious. Suppose the governing equation is

$$\frac{\partial c}{\partial t} + x\,\frac{\partial c}{\partial x} = 0, \tag{112}$$

which is linear, but with a variable coefficient, x. The IC is presumed to be

$$c(x, t=0) = f(x_o), \tag{113}$$

where the function f can be left unspecified. The characteristic equations of (112) are

$$\frac{dt_\gamma}{ds} = 1, \quad \text{and} \quad \frac{dx_\gamma}{ds} = x. \tag{114}$$

Assuming IC

$$t_\gamma(s=0) = 0 \quad \text{and} \quad x_\gamma(s=0) = x_o,$$

then the characteristic curves are

$$t_\gamma = s, \quad \text{and} \quad x_\gamma = x_o \exp(s) = x_o \exp(t_\gamma). \tag{115}$$

These characteristics make a fan shape centered on $x = 0$ that widens exponentially with time. The solution comes from eliminating x_o between (113) and (114), to find

$$c(x,\, t) = f(x \exp(t)), \tag{116}$$

which you should check by substitution into the governing equation (112).

If the initial tracer distribution is a Gaussian, then

$$c(x,\, t) = \exp(-(x \exp(t))^2/2), \tag{117}$$

which is not so easy to intuit. However, if the fan-shaped characteristics of Fig. (17), left, made sense, and if you can imagine propagating $c(x_o) = \exp(-x_o^2/2)$ along those characterstics, then it is pretty clear that the eolsution will be (117) and Fig. (17), right. This is an example, and there are many others, wherein the characteristics of the MoC are a very effective aid to visualizing (intuiting) a solution.

A surprising property of this solution is that the amount of tracer increases exponentially in time, evident in Fig. (18). To understand how this happens, it is helpful to rewrite (112) in conservation form,

$$\frac{\partial c}{\partial t} + \frac{\partial(\text{flux of c})}{\partial x} = 0,$$

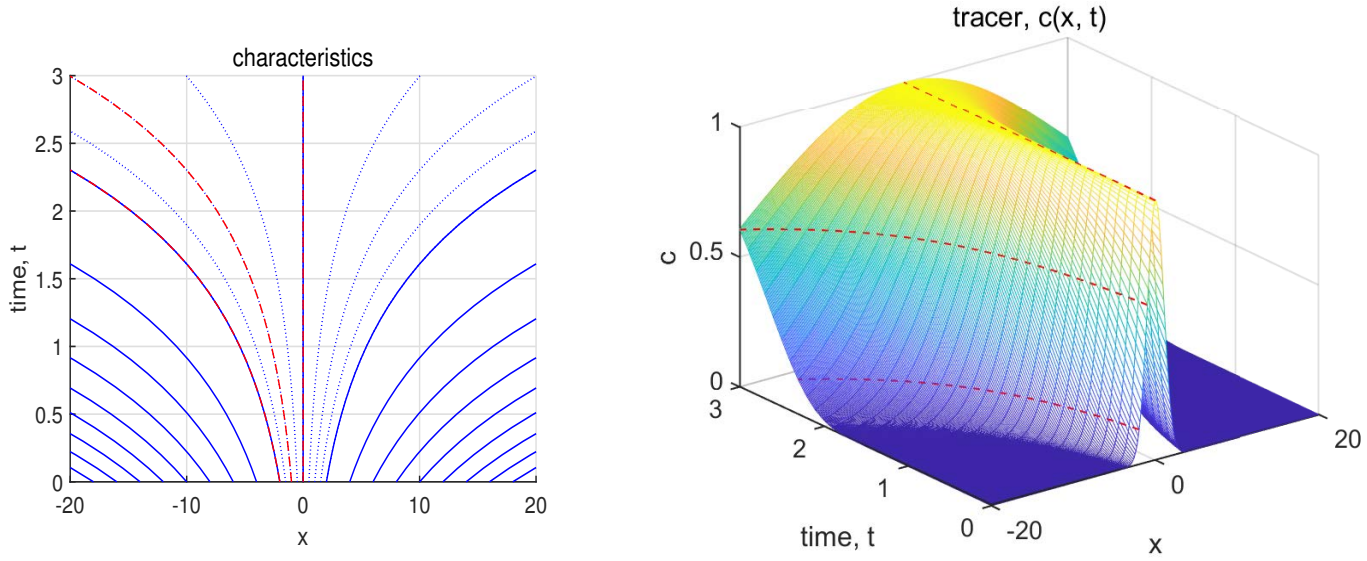

Figure 17: **(left)** Characteristic curves of the governing equation (112). The red dashed curves are for $x_o = 0, -1, -2$. **(right)**. The tracer $c(x,t)$ from (112) and (113) asuming that $f(x_o) = exp(-x_o^2/2)$. The tracer is constant along the characteristics at left, including the three red, dashed lines which correspond to those at left. Notice that the area under the curve $c(x)$ increases with time, and thus the total amount of tracer is not conserved.

as far as possible. Consider that

$$\frac{\partial(xc)}{\partial x} = c + x\frac{\partial c}{\partial x},$$

and so if c is added to both sides of the governing equation (112) and then

$$\frac{\partial c}{\partial t} + \frac{\partial(xc)}{\partial x} = c. \tag{118}$$

The lefthand side is in conservation form, but now there is a source term c on the righthand side. It might not have been obvious looking at the original PDE (112), but from its equivalent (118), it is apparent that the net amount of tracer should increase with time, as indeed it does. Moreover, the rate of increase is proportional to c, i.e., exponential.

4.4.3 When the current is advected

Now consider a problem in which the tracer is the current u,

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{119}$$

The initial condition is presumed to be a Gaussian hump,

$$u(x, t = 0) = U\exp(-x_o^2/2), \tag{120}$$

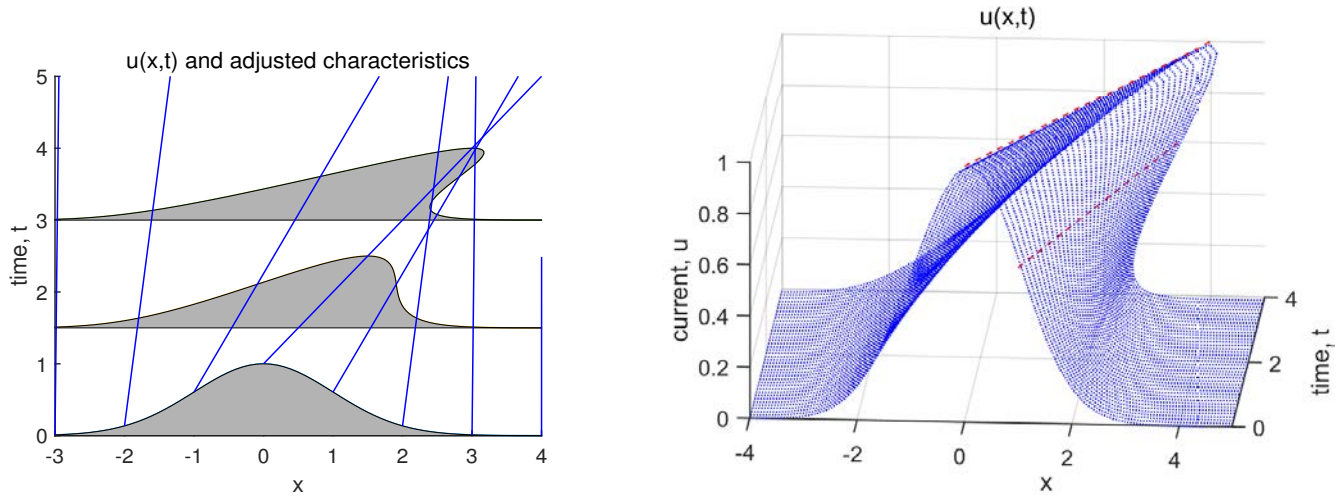

Figure 18: **(left)** The characteristics of the flow governed by Eqs. (119) and (120). Because the initial current profile varies with x (a Gaussian) the characteristics do not have a uniform slope. **(right)**. The current profile, $u(x,t)$ as a surface in three-dimensions. The Gaussian hump propagates to the right, and the peak of u moves the fastest. Consequently the slope on the rightward side of the peak increases until $t = \exp(1/2) \approx 1.7$ when this $u(x,t)$ becomes multivalued, which is unphysical. The elementary MoC solution (122) no longer applies, and the subsequent evolution has to account for the formation of a shock across which the current may be discontinuous but still momentum-conserving.

with U the amplitude (not the uniform value, as in Sec. 4.2). This governing equation may be rewritten in a conservation form

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u^2/2}{\partial x} = 0.$$

The governing equation (119), often called the inviscid Burgers equation, is not linear because the advection term is the product of two unknowns, u and $\partial u/\partial x$, and so the wide array of methods that are available to solve linear PDEs will not be applicable. This governing equation is linear in its first order, partial derivatives, however, and well-suited for treatment via the MoC.

The current u is constant along characteristics,

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} = 0$$

that are the solutions of

$$\frac{dt_\gamma}{ds} = 1, \quad \text{and} \quad \frac{dx_\gamma}{ds} = u(x_o).$$

The characteristics are then

$$t_\gamma = s, \quad \text{and} \quad x_\gamma = x_o + U exp(-x_o^2/2)\, s,$$

with an explicit solution (eliminate s)

$$x_\gamma = x_o + U exp(-x_o^2/2)\, t_\gamma. \tag{121}$$

These characteristics are straight lines, Fig. (18), left, since the speed u is constant along characteristics. But a significant difference from the elementary problem of Sec. 4.2 is that the slope of the characteristics will vary with x_o. Far from the origin, the characteristic lines are nearly vertical in the x,t plane, while close to the origin the characteristics have a minimum slope and thus a maximum dx/dt and u.

The solution may be found by eliminating x_o from (120) and (121),

$$u(x,t) \;=\; U\exp\left(-(x-u(x,t)t)^2/2\right), \tag{122}$$

which is not separable into an explicit solution (u isolated on one side of the equation). This is typical of quasi-linear problems such as Burgers equation. Because the governing inviscid Burgers equation can be written in conservation form, and because the distribution of u is bounded, the domain-wide integral of u is conserved.

The solution (122) is not easy to see until it is graphed, Fig. (18), right, and even then the interpretation is aided considerably by reference to the characteristics. The most rapidly moving part of the hump that begins near the origin ($a = 0$) will sooner or later overtake the slower moving part that began at larger x (this coarse description is refined in Problem 5) below). After some elapsed time, the charactersistics will cross, and u at that point will have an infinite derivative, $\partial u/\partial x$, Fig. (18), right. If the method of characteristics is continued, the solution for u will then appear to be triple-valued at that (x,t) and beyond. In the case that u is the current, a multi-valued solution makes no physical sense and has to be rejected. Evidently the governing equation (119) has omitted some physical process(es), e.g., diffusion or viscosity, that will become important when the derivative $\partial/\partial x$ becomes large. Even with diffusion present, the derivative may, nevertheless, become very large and the flow said to form a shock wave, across which the current is nearly discontinuous. The conservation of momentum should hold regardless of the details of the field, and the subsequent motion of a shock wave can be determined from fundamental principles.

4.5 Key ideas

1. An elementary initial value problem for a passive tracer in an ideal fluid flow may be solved in either a Lagrangian or an Eulerian system. The Lagrangian solution comes in two parts, tracer concentration on moving parcels (conserved at the initial value if the flow is adiabatic) and the position of those parcels. The Eulerian solution is complete in one piece, a tracer field that depends upon the field coordinates, space and time.

2. The two-part Lagrangian solution may be readily transformed into the Eulerian solution by eliminating reference to the initial position. The Eulerian solution for tracer concentration alone is not so readily transformed to the Lagrangian solution.

3. The method of characteristics (MoC) is applicable to the solution of first order PDEs, which arise in many contexts besides fluid mechanics. The tracer is represented on a field, and hence the MoC is an Eulerian method. The starting point is the recognition that the tracer balance will be simplest when evaluated along the characteristic curves of the governing PDE. The characteristics may be

computed from the coefficients of the partial derivative terms, and if the PDE is an advection equation, the characteristics are parcel trajectories.

4. If the flow is adiabatic (no source or diffusion) then the tracer will be constant along a characteristic. If a source is present, then it may be accounted by another ODE that solves for the tracer along the characteristics, i.e., along parcel trajectories.

5. The interpretation of an Eulerian solution can borrow Lagrangian concepts, even when the governing PDE is not an advection equation.

4.6 Problems

1) The method of characteristics makes important use of the idea of parameterized curves. The independent variable of these curves is often represented by s, which can seem a little mysterious because it need not be defined precisely. This problem includes a brief background on parameterized curves, and then offers a short quiz that should help refresh your memory of this occasionally very useful device.

The points $(x,\ y)$ that lie on a circle of radius r are related by the familiar

$$x^2 + y^2 = r^2. \tag{123}$$

Suppose that you need to write y as an explicit function of x for use in some later calculation. The algebra is trivial, $y = \pm\sqrt{r^2 - x^2}$, and the meaning is clear. However, this has not given us a function because one value of x maps into two values of y and vice-versa (which is, of course, a fundamental property of a circle). One work-around is to define two functions, say $y_+ = +\sqrt{r^2 - x^2}$ and $y_- = -\sqrt{r^2 - x^2}$, and then check each time to insure the correct choice. That is inelegant and can be inefficient.

Alternatively, consider a parameterized version of (123), say

$$x_\gamma = r\cos(s) \quad \text{and} \quad y_\gamma = r\sin(s), \tag{124}$$

where s is the independent variable, and in this context often said to be the 'parameter' of this new version of a circle. This parameterized circle is defined via explicit functions, although two of them, and now with this interloper, s. Are there advantages to this? Here are some questions that will help you decide.

(i) Over what range should s be varied in (124) in order to fill out a complete circle?

(ii) Show that the slope of the tangent to the curve is $(dy_\gamma/ds)/(dx_\gamma/ds)$.

(iii) Suppose the same range of s as in 1), but now take the parameterization to be $x_\gamma = r\cos(2s)$ and $y_\gamma = r\sin(2s)$. What changed? Does the slope formula still work? Suppose that s is proportional to time; how about the rate, $\partial x_\gamma/\partial t$?

(iv) Now try $x_\gamma = r\cos(s^2)$ and $y_\gamma = r\sin(s^2)$. Same questions.

(v) It is beginning to seem as if s can be almost anything, so let's try an experiment with the relationship between s, x_γ and y_γ, say $x_\gamma = r\cos(s)$ and $y_\gamma = r\sin(2s)$ and then $x_\gamma = r\cos(s)$ and $y_\gamma = r\sin(s^2)$. Do these parameterize a circle?

(vi) We didn't say what kind of numbers s was meant to be, so you probably assumed the reals. Suppose instead that s is the integers, or how about the Fibonacci sequence?

(vii) So, after all of this ..... what is s ? 

2) Given the Lagrangian solution Eqs. (94) and (95), compute the corresponding Eulerian solution. Could you do this absent either (94) or (95)?

3) Compute the Lagrangian solution for the case of a source or sink, Eq. (109) in Sec. 4.4.1.

4) An initial value problem for a tracer $c(x,t)$ that you can solve via the method of characteristcs:

$$\frac{\partial c}{\partial t} + U\frac{\partial u}{\partial x} = 1, \quad \text{and} \quad c(x,t=0) = cos(x),$$

where U is a constant. (Major hint: $c(x,t) = t + cos(x-Ut)$). For an excellent web-based resource that will help you with this problem, see https://www.youtube.com/watch?v=tNP286WZw3o A Matlab script that implements the method of charcteristics in symbolic math is https://www2.whoi.edu/staff/jprice/wp-content/uploads/sites/199/2023/10/MoCsymbolic.txt

5) Crossing characteristics. We saw in Sec. 4.4.3 that characteristic curves will eventually cross when (as usually occurs) the current varies in space. Let's find out when that will happen for the problem considered there. It is helpful to think of the motion along characteristics as the motion of fluid parcels whose x positions are given by $x = x_o + \exp(-x_o^2/2)$. A 'collision' will occur when $\partial x(x_o,t)/\partial x_o = 0$, and the separation between parcels vanishes. Can you show that the first time of such a collision is $t_{coll} = \exp(1/2)$ and the location is $x_{coll} = 2$, cf. Fig. (18). Collisions are especially noteworthy, but not the only phenomena illuminated by the characterstics; e.g., what happens on the left side of the current maximum?

6) Given a first order, nonlinear PDE

$$\frac{\partial u}{\partial t} + h(u)\frac{\partial u}{\partial x} = 0$$

with IC

$$u(x,\, t=0) = \Psi(x).$$

Verify by direct substitution that an implicit solution is

$$u(x,\, t) = \Psi(x - h(u)\, t).$$

5 Coming up in Part 2

A complete model of a fluid flow will likely require the Eulerian mass and momentum equations derived in Sec. 3, together with a state equation. The definition of a specific problem requires a number of additional steps including simplification where appropriate, and careful consideration of the boundary and initial conditions. Finding and exploring the solutions of such a system will often require a great deal of time and effort, and frequently an appeal to numerical and experimental methods. Most of your research of fluid mechanics will be devoted to just that kind of task.

The scope and goals of the Part 2 essay start with the (generous) assumption that you have generated an analytic or a numerical solution of the velocity field. The question addressed in Part 2 is how to analyze these solutions for properties of interest, especially parcel trajectories. Thus the direction of Part 2 is Eulerian to Lagrangian.

6 Appendices

6.1 Physical properties of fluids and solids

Classical fluid mechanics, like classical thermodynamics, is concerned with macroscopic phenomena (bulk properties) rather than microscopic (molecular-scale) phenomena. In fact, the molecular makeup of a fluid will be studiously ignored in all that follows, and the crucially important physical properties of a fluid, e.g., its mass density, ρ, heat capacity, Cp, among others, must be provided from outside of this theory, Table (1). It will be assumed that these physical properties, along with flow properties, e.g., the pressure, P, velocity, $\mathbf{V}$, temperature, T, etc., are in principle definable at every point in space, as if the fluid was a smoothly varying continuum, rather than a swarm of very fine, discrete particles (molecules).

The space occupied by the material will be called the domain. Solids are materials that have a more or less intrinsic configuration or shape and do not conform to their domain under nominal conditions. Fluids do not have an intrinsic shape; gases are fluids that will completely fill their domain (or container) and liquids are fluids that form a free surface in the presence of gravity.

An important property of any material is its response to an applied force, Fig. (19). If the force on the face of a cube, say, is proportional to the area of the face, as will often be the case, then it is appropriate to consider the force per unit area, called the stress, and represented by the symbol S if a scalar, by $\mathbf{S}$ if a three component stress vector, and by $\mathbb{S}$ when considering a nine component stress tensor. The latter is introduced here briefly and in more detail in Sec. 3.4. The SI units of stress are Newtons per meter squared, which is commonly represented by a derived unit, the Pascal, or Pa. Why there is a stress and how the stress is related to the physical properties and the motion of the material are questions of first importance in many investigations, but for now the stress is presumed given.

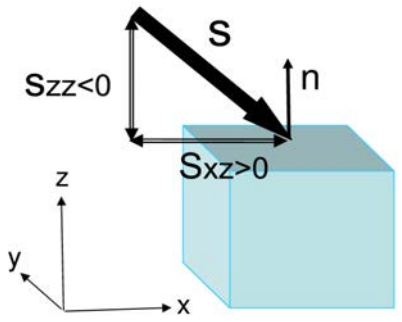

Figure 19: An orthogonal triad of Cartesian unit vectors and a small cube of material. The surrounding material is presumed to exert a stress, **S**, upon the face of the cube that is normal to the z axis. The outward-directed unit normal of this face is $\mathbf{n} = \mathbf{e_z}$. To manipulate the stress vector it will usually be necessary to resolve it into components: Szz is the projection of **S** onto the $\mathbf{e_z}$ unit vector and is negative, and S_{xz} is the projection of **S** onto the $\mathbf{e_x}$ unit vector and is positive. Thus the first subscript on S indicates the direction of the stress component and the second subscript indicates the orientation of the face upon which it acts. This ordering of the subscripts is a convention, and it is not uncommon to see this reversed.

The component of stress that is normal to the upper surface of the material in Fig. (19) is denoted S_{zz}. A normal stress can be either a compression, if $S_{zz} \leq 0$, as implied in Fig. (20), or a tension, if $S_{zz} \geq 0$. The most important compressive normal stress is almost always due to pressure rather than to viscous effects, and when the discussion is limited to compressive normal stress only, S_{zz} is identified as the pressure.

6.1.1 In response to a normal stress, liquids and solids are not very different

Every material will undergo some volume change as the ambient pressure is increased or decreased, though the amount varies quite widely from gases to liquids and solids. To make a quantitative measure of the volume change, let P_0 be the nominal pressure and h_0 the initial thickness of the fluid sample; denote the pressure change by δP and the resulting thickness change by δh. The normalized change in thickness, $\delta h/h_0$, is called the linear deformation (linear in this case meaning that the displacement is in line with the stress). The linear deformation is of special significance in this one-dimensional configuration because the volume change is equal to the linear deformation, $\delta V = V_0 \delta h/h_0$ (in a two- or three-dimensional fluid this need not be the case, Section 6.3). The mass of material, $M = \rho V$, is not affected by pressure changes and hence the mass density, $\rho = M/V$, will change inversely with the linear deformation;

$$\frac{\delta\rho}{\rho_0} = -\frac{\delta V}{V_0} = -\frac{\delta h}{h_0}, \tag{125}$$

where δ is a small change, $\delta \ll 1$. Assuming that the dependence of thickness change upon pressure can be observed in the laboratory, then $\delta h = \delta h(P_0, \delta P)$ together with Eq. (125) are the rudiments of an equation of state, the functional relationship between density, pressure and temperature, $\rho = \rho(P,T)$ or equivalently, $P = P(\rho,T)$, with T the absolute temperature in Kelvin. The archetype of an equation of state is that of an ideal gas, $PV = nRT$ where n is the number of moles of the gas and $R = 8.31$ Joule moles^{-1} K^{-1} is the universal gas constant.

Some physical properties of air, sea and land (granite)

	density ρ, kg m^{-3}	heat capacity Cp, J kg^{-1} C^{-1}	bulk modulus B, Pa	sound speed c, m s^{-1}	shear modulus K, Pa	viscosity ν, Pa s
air	1.2	1000	1.3×10^5	330	na	18×10^{-6}
sea water	1025	4000	2.2×10^9	1500	na	1×10^{-3}
granite	2800	2800	4×10^{10}	5950	2×10^{10}	$\geq 10^{22}$

Table 1: Approximate, nominal values of some thermodynamic variables that are required to characterize materials to be described by a continuum theory. These important data must be derived from laboratory studies. For air, the values are at standard temperature, 0 C, and nominal atmospheric pressure, 10^5 Pa. The bulk modulus shown here is for adiabatic compression; under an isothermal compression the value for air is about 30% smaller; the values are nearly identical for liquids and solids. na is not applicable. The viscosity of granite is temperature-dependent; granite is brittle at low temperatures, but appears to flow as a highly viscous material at temperatures above a few hundred C.

An important class of phenomenon may be described by a reduced equation of state having state variables density and pressure alone,

$$\rho = \rho(P), \quad \text{or equivalently,} \quad P = P(\rho). \tag{126}$$

It can be presumed that ρ is a monotonic function of P and hence that $P(\rho)$ should be a well-defined function of the density. A fluid described by Eq. (126) is said to be 'barotropic' in that the gradient of density will be everywhere parallel to the gradient of pressure, $\nabla\rho = (\partial\rho/\partial P)\nabla P$, and hence surfaces of constant density will be parallel to surfaces of constant pressure.

The stiffness or inverse compressibility of the material is

$$B = \frac{S_{zz}}{\delta h/h} = -V_0\frac{\delta P}{\delta V} = \rho_0\frac{\delta P}{\delta\rho}, \tag{127}$$

called the bulk modulus. Notice that B has the units of stress or pressure, Pa, and is much like a normalized spring constant; B times the normalized linear strain (or volume change or density change) gives the resulting pressure change. The numerical value of B is the pressure increase required to compress the volume by 100% of V_0. Of course, a complete compression of that sort does not happen outside of black holes, and the bulk modulus should be regarded as the first derivative of the state equation, accurate for small changes around the ambient pressure, P_0. Gases are readily compressed; a pressure increase $\delta P = 10^4$ Pa, which is 10% above nominal atmospheric pressure, will cause an air sample to compress by about $B^{-1}10^4 Pa = \delta V/V_o = 7\%$ under adiabatic conditions. Most liquids are quite resistant to compressive stress, e.g., for water, $B = 2.2\times10^9$ Pa, which is less than but comparable to the bulk modulus of a very stiff solid, granite (Table 1). Thus the otherwise crushing pressure in the abyssal ocean, up to about 1000 times atmospheric pressure in the deepest trench, has a rather small effect upon sea water, compressing it and raising the density by only about five percent above sea level values. Water is stiff enough and pressure changes associated with geophysical flows small enough that

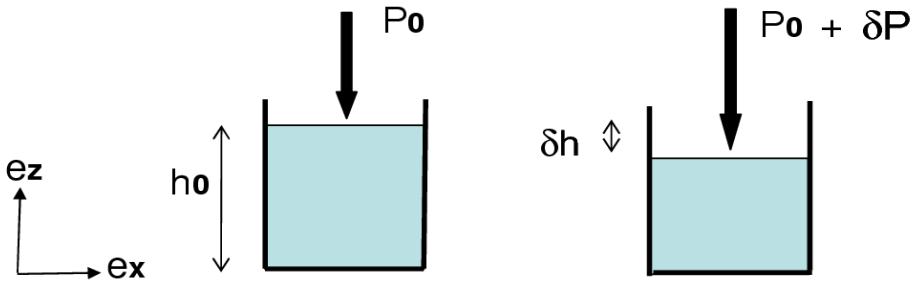

Figure 20: A solid or fluid sample confined within a piston has a thickness h_0 at the ambient pressure P_0. If the pressure is increased by an amount δP, the material will be compressed by the amount δh and the volume decreased in proportion. The work done during this compression will raise the temperature of the sample, perhaps quite a lot if the material is a gas, and it becomes important to define whether the sidewalls allow heat flux into the surroundings (isothermal compression) or not (adiabatic compression); the B in Table 1 is the latter.

for many purposes water may be idealized as an incompressible fluid, as if B was infinite. Surprisingly, in many circumstances the same is true for air.

The first several physical properties listed in Table 1 suggest that water has more in common with granite than with air, our other fluid. The character of fluids becomes evident in their response to anything besides a compressive normal stress. Fluids are qualitatively different from solids in their response to a tensile normal stress, i.e., $S_{zz} \geq 0$, which is resisted by many solid materials, especially metals, with almost the same strength that they exhibit to compression. In contrast, gases do not resist tensile stress at all, while liquids do so only very, very weakly when compared with their resistance to compression. Thus if a fluid volume is compressed along one dimension but is free to expand in a second, orthogonal, direction (which the one-dimensional fluid confined in a piston, Fig. (20), can not, of course) then the volume may remain nearly constant though the fluid may undergo significant linear deformation, i.e., a compression and a nearly compensating expansion in an orthogonal direction.

6.1.2 In response to shear stress, solids deform and fluids will *flow*, often in very complex ways

A stress that is parallel to (in the plane of) the surface that receives the stress is called a 'shear' stress.[37] A shear stress that is in the x direction and applied to the upward face of the cube in Fig. (19) would be labeled S_{xz} and a shear stress in the y-direction, S_{yz}. A measure of a material's response to a steady shear stress is the shear deformation, r/h, where r is the steady (equilibrium) sideways displacement of the face that receives the shear stress and h is the column thickness (Fig. 21, and note that the cube of material is presumed to be stuck to the lower surface). The corresponding stiffness for shear stress, or shear

[37] The word *shear* has an origin in the Middle English *scheren*, which means to cut with a pair of sliding blades. A velocity shear is a spatial variation of the velocity in a direction that is perpendicular to the velocity vector.

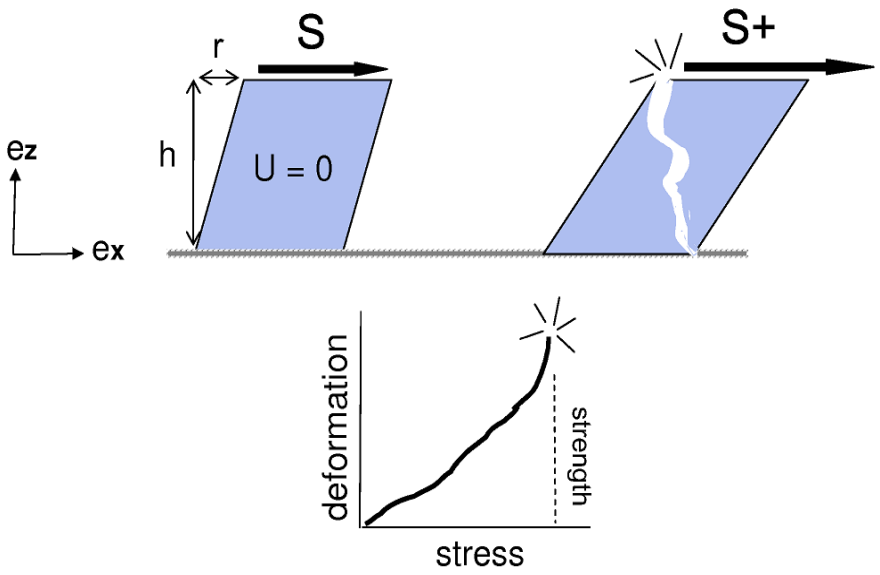

Figure 21: A vector stress, **S**, is imposed upon the upper face of a cube of solid material that is attached to a lower surface. Given the orientation of this face with respect to the unit vectors, this stress can also be represented by a single component, S_{xz}, of the stress tensor (Section 2.2.1). For small values of the stress, a solid will come to a static equilibrium in which an elastic restoring force balances the shear stress. The shear deformation (also called the shear 'strain') may be measured as r/h for small angles. It is fairly common that homogeneous materials exhibit a roughly linear stress/deformation relationship for small deformations. But if the stress exceeds the strength of the material, a solid may break, an irreversible transition. Just before that stage is reached the stress/deformation ratio is likely to decrease.

modulus, is then defined as

$$K = \frac{S_{xz}}{r/h}, \tag{128}$$

which has units of pressure. The magnitude of K is the shear stress required to achieve a shear deformation of $r/h = 1$, which is past the breaking point of most solid materials. For many solids the shear modulus is comparable to the bulk modulus (Table 1).[38]

Fluids are qualitatively different from solids in their response to a shear stress. Ordinary fluids such as air and water have no intrinsic configuration, and hence fluids do not develop a restoring force that can provide a static balance to a shear stress.[39] There is no volume change associated with a pure shear

[38] The distinction between solid and fluid seems clear enough when considering ordinary times and forces. But materials that may appear unequivocally solid when observed for a few minutes may be observed to flow, perhaps only very, very slowly, when observed over many days or millenia. Glaciers are a very important natural example, and see also the pitch drop experiment: https://smp.uq.edu.au/pitch-drop-experiment.

[39] There is no volume change associated with a pure shear deformation and thus no coupling to the bulk modulus. There does occur a significant linear deformation, compression and expansion, in certain directions examined in a later section, 6.3. While fluids have no *intrinsic* restoring forces or equilibrium configuration, nevertheless, there are important restoring forces set up within fluids in the presence of an acceleration field. Most notably, gravity will tend to restore a displaced free surface back towards level. Earth's rotation also endows the atmosphere and oceans with something closely akin to angular momen-

deformation and thus no coupling to the bulk modulus. Hence, there is no meaningful shear modulus for a fluid since r/h will not be steady. Rather, the distinguishing physical property of a fluid is that it will move or 'flow'[40] in response to a shear stress, and moreover, a fluid will continue to flow so long as a shear stress is present.

When the shear stress is held steady, and assuming that the geometry does not interfere, the shear deformation rate, $h^{-1}(dr/dt)$, may also be steady or have a meaningful time-average. In analogy with the shear modulus, a generalized viscosity, Υ, may be defined as the ratio of the measured shear stress to the shear deformation rate,

$$\Upsilon = \frac{S_{xz}}{h^{-1}dr/dt}. \tag{129}$$

This ratio of shear stress to shear deformation rate will depend upon the kind of fluid material and also upon the flow itself, i.e., the speed, $U = dr/dt$ of the upper moving surface and the column thickness, h. This generalized viscosity times a unit, overall velocity shear $U(z=h)/h = h^{-1}(dr/dt) = 1\ \mathrm{s}^{-1}$ is the shear stress required to produce the unit velocity shear.

Laminar flow at small Reynolds number: If the flow depicted in Fig. 22 is set up carefully, it may happen that the fluid velocity U will be steady, with velocity vectors lying smoothly, one on top of another, in layers or 'laminar' flow (the upper left of Fig. 22). The ratio of stress to shear then defines a fluid property

$$\text{viscosity:} \qquad \boxed{\nu = \frac{S_{xz}}{\partial U/\partial z}} \tag{130}$$

or sometimes dynamic viscosity.

Newtonian fluids, air and water: Fluids for which the viscosity in laminar flow is a thermodynamic property of the fluid alone and not dependent upon the shear stress magnitude are dubbed 'Newtonian' fluids, in recognition of Isaac Newton's insightful analysis of frictional effects in fluid flow. Air and water are found to be Newtonian fluids to an excellent approximation.

If the fluid is Newtonian, then it is found empirically that the conditions for laminar flow include that a nondimensional parameter called the

$$\text{Reynolds number:} \qquad \boxed{Re = \frac{\rho U h}{\nu}} \tag{131}$$

will satisfy the inequality $Re \leq 400$, where U is the speed of the upper (moving) surface relative to the lower, fixed, no-slip surface. The Reynolds number arises several times in this essay, and will be discussed in more detail in a later section, 3.5.4. If water, then in practice this means that the speed must be very low or the column thickness very small. The laminar flow velocity $U(z)$ of a Newtonian fluid will vary linearly with z and the velocity shear at each point in z will then be equal to the overall shear deformation rate, $\partial U/\partial z = h^{-1}(dr/dt)$, the particular laminar flow sketched in Fig. 22 upper left.

tum that provides a restoring tendency for horizontal displacements; the oscillatory wave motion seen in the cover graphic is an example.

[40] Fluid and flow have a nearly common eytemology in the Latin *fluidus* and *fluere*.

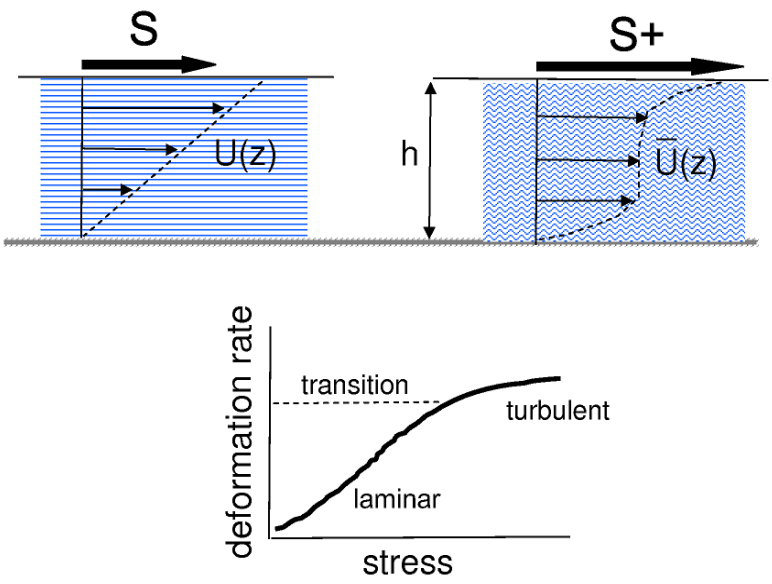

Figure 22: A vector stress, **S**, is imposed upon the upper face of a cube of fluid material that is sitting on a no-slip lower surface. Since we are considering only the z-dependence of the flow, it is implicit that the fluid and the stress are uniform in the horizontal. The response of a fluid to a shear stress is quite different from that of a solid in as much as a fluid has no intrinsic shape and so develops no elastic restoring force in response to a deformation. Instead, an ordinary fluid will move or flow so long as a shear stress is imposed and so the relevant kinematic variable is the shear deformation rate. For small values of the stress and assuming a Newtonian fluid, the fluid velocity, $U(z)$, may come into a laminar and steady state with a uniform vertical shear, $\partial U/\partial z = U(h)/h = constant = S_{xz}/\nu$, that can be readily observed and used to infer the fluid viscosity, ν, given the measured stress. For larger values of stress (right side) the flow may undergo a reversible transition to a turbulent state in which the fluid velocity is two or three-dimensional and unsteady despite that the stress is steady. The time average velocity $\overline{U(z)}$ is likely to be well-defined provided the external conditions are held constant. In this turbulent flow state, the time-averaged shear $\partial\overline{U}/\partial z$ will vary with z, being larger near the boundaries. The shear stress and the time-averaged overall deformation rate, $\overline{U(h)}/h$, are not related by a constant viscosity as obtains in the laminar flow regime, and across the turbulent transition the stress/deformation rate ratio will increase.

Assuming that the fluid viscosity and its dependence upon temperature, density, etc. are known, then the relationship Eq. (130) between viscosity, stress and velocity shear may just as well be turned around and used to estimate the viscous shear stress from a given velocity shear. This is the way that viscous shear stress will be incorporated into the momentum balance of a fluid parcel (Section 3.4.3). It is important to keep in mind that Eq. (130) is not an identity or a universal, physical law, but rather a contingent experimental law that applies only for laminar, steady flow. If instead the fluid velocity is unsteady and two- or three dimensional, i.e., turbulent and which is much more common, then for a given upper surface speed $U(h)$, the shear stress will be larger, and sometimes quite a lot larger, than the laminar value predicted by Eq. (130) (Figure 22).[41] Evidently then, Eq. (130) has to be accompanied by

[41] Viscosity and turbulence can in some limited respects mimic one another; a given stress and velocity shear can be consistent with either a large viscosity in laminar flow, or, a smaller viscosity (and thus higher Reynolds number) in tur-

Eq. (131) along with a description of the geometry of the flow, i.e., that h is the distance between parallel planes (and not the distance from one plate or the diameter of a pipe, for example). In most geophysical flows the equivalent Reynolds number is enormously larger than the upper limit for laminar flow indicated by Eq. (131) and consequently geophysical flows are seldom laminar and steady, but are much more likely to be turbulent and unsteady. Thus, for a given velocity shear, it very commonly happens that properties of the flow, rather than the physical properties of the fluid alone, determine the stress.

6.2 Composite functions and inverse functions

6.2.1 Functions of functions

Composite functions are, essentially, functions of functions. The formalism of composite functions is relevant on several occasions, including that the transformation of velocity from a Lagrangian to an Eulerian system amounts to the construction of a composite function, the velocity as a function of position.

This appendix aims to review some of the salient features of composite and inverse functions using simple examples. Let f by a function of a single variable, say

$$f(y) = 1 + y/3, \tag{132}$$

in which case the notation $f(y)$ means — take the variable found in the argument, multiply by $1/3$ and then add 1 (simple indeed). An equation of the sort Eq. (132) does not by itself make it clear whether we mean to define the function f, or whether we intend to evaluate $f(y)$ by assigning a value to y. In this context it is clear that this defines f, since y serves only to hold a place, i.e., it is a dummy variable that could just as well have be written x or z.

Now suppose that y can be or must be regarded as a function of an independent variable x,

$$y = g(x),$$

and for example,

$$g(x) \; = \; x^2, \tag{133}$$

with x real and $x > 0$. Restricting the domain of x and y insures that the inverse of g, written g^I, exists and will be used below.

A composite function, h, may be constructed from f and g by taking g to be the argument for f, i.e.,

$$h \; = \; f(g). \tag{134}$$

bulent flow. The pioneering investigators of liquid helium assumed that the flow in the very small laboratory apparatus used to estimate viscosity must be laminar, when in fact it was turbulent. This delayed the recognition that superfluid helium has a nearly vanishing viscosity (A. Griffin, Superfluidity: a new state of matter. In *A Century of Nature*. Ed. by L. Garwin and T. Lincoln. The Univ. of Chicago Press, 2003.) An excellent introduction to modern experimental research on turbulence including some Lagrangian aspects is by R. Ecke, The turbulence problem, available online at http://library.lanl.gov/cgi-bin/getfile?01057083.pdf

The function g inside the parentheses is termed the inner function, and f is termed the outer function. Given the present example (132) and (133), the composite function is

$$h(x) \;=\; f(g(x)) = f(x^2) = 1 + x^2/3, \tag{135}$$

plotted as the blue line of (Fig. 138). Thus when we discuss a composite function it is implicit that there are three functions in play: an outer function f, an inner function g, and the composite function, $h = f(g)$. Notice that the composite function depends upon the dummy variable x, i.e., the independent variable of a composite function is the independent variable of the inner function.

In general,

$$h(x) \neq f(y),$$

where the intent here is that h is to be evaluated at x and f is to be evaluated at y, with as yet no restriction placed upon x or y. However, it is true that

$$h(x) = f(y)\,|_{y=g(x)}, \tag{136}$$

where $(\;)\,|_{y=y(x)}$ means that the argument of f is equal to the inner function evaluated at the x that appears as the argument on the left hand side. In other words, it is true that $h(x) = f(y)$ just in case the values of x and y are related by the inner function. With a little reflection this seems fairly obvious, but an equivalent way to state this is not quite so transparent, namely

$$h(x)\,|_{x=g^I(y)} \;= f(y), \tag{137}$$

where g^I indicates the inverse of the inner function, g. In the example noted above,

$$g^I(y) = y^{1/2},$$

the positive square root of y, and sure enough,

$$h(y^{1/2}) \;=\; 1 \,+\, y/3 \;=\; f(y).$$

The relation (137) is exactly the form used to state the FPK, Eq. (4), the definition of Eulerian velocity:

the Eulerian velocity $\mathbf{V} \rightarrow h$, the composite function,
the field coordinate, $\mathbf{X} \rightarrow x$, the argument of the composite function
the Lagrangian velocity $\tilde{\mathbf{V}} \rightarrow f$, the outer function,
the trajectory inverse, $\mathbf{A}(\tilde{\mathbf{X}}) \rightarrow g(x)$, the inner function, and
the material coordinate, $\tilde{\mathbf{X}} \rightarrow y$, the argument of the outer function.

Thus when we start with a Lagrangian perspective, presuming to know the Lagrangian trajectories, their inverse and the Lagrangian velocity, then the Eulerian velocity field can be constructed as a composite function made up from these Lagrangian data — no derivatives or integrations required.

6.2.2 Inverse functions and their derivatives

A somewhat general definition of an inverse function g^I is

$$g^I(g(x)) \;=\; g(g^I(x)) \;=\; x. \tag{138}$$

Thus the inverse function undoes the function (nothing surprising in that). The relation (138) is especially easy to apply to the example $g(x) = x^2$ suggested here. Since the domain of x was restricted to the positive, real numbers, the inverse function in this case is just

$$g^I(x) \;=\; x^{1/2},$$

the positive square root.

The inverse of the slightly more complex composite function (135) is best found with a more systematic procedure.[42] Let $w \;=\; h(x) = 1 + x^2/3$. Now solve this for $x \;=\; \sqrt{(3w-3)}$. Switch the places (labels) of x and w, and take this new x to be the argument of the putative inverse function,

$$h^I(x) \;=\; \sqrt{(3x \;-\; 3)}, \tag{139}$$

the red line of Fig. (138). You can check that (139) is indeed the inverse function of (135) by verifying

$$h^I(h(x)) \;=\; x. \tag{140}$$

The derivative of an inverse function bares a simple and useful relationship to the derivative of the function itself. Let's show this for the composite function, h and it's inverse h^I noted just above. Take the derivative of the left and right sides of (140) using the chain rule to find

$$\frac{dh^I(h)}{dh}\,\frac{dh}{dx} \;=\; \frac{dx}{dx} \;=\; 1,$$

and so

$$\frac{dh^I(h)}{dh} \;=\; 1/\frac{dh}{dx}. \tag{141}$$

The left side derivative d/dh might look a little unusual but is easy to evaluate; take the derivative of $h^I(w)$ with respect to a dummy argument, w, i.e., $dh^I/dw \;=\; (3/2)/\sqrt{3w \;-\; 3}$, and then evaluate this at $w \;=\; h(x) \;=\; 1 \;+\; x^2/3$, to find $dh^I(h)/dh \;=\; (3/2)\,x^{-1}$, the reciprocal of dh/dx of (135). Thus the derivative of the inverse of a function is the reciprocal of the derivative of the function (this is easier to see in the example of Fig. (23) than it is to say). Eq. (141) is a very useful result when it comes time to convert derivatives from one coordinate system to another in Sec. 2.2, and on.

[42] An excellent refresher on the procedure for finding the inverse of a function may be found online by a search for 'How to find the inverse of a function (NancyPi)'.

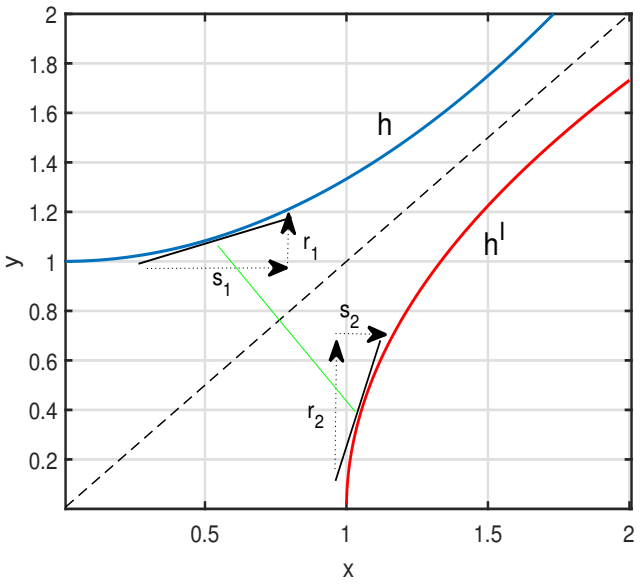

Figure 23: The function $h(x) = 1 + x^2/3$ is the blue line, and its inverse, $h^I(x) = (3x-3)^{1/2}$, is the red line. The inverse, h^I, is the reflection of h across $y = x$, the dashed, black line. Notice that the slopes of h and h^I are reciprocals when they are evaluated at points that are reflections across $x = y$, e.g., the thin green line connects the point $(x,\ y) = (0.5,\ h(0.5))$ with its reflection, $(x,\ y) = (h(0.5),\ h^I(h(0.5))$. By inspection, $r_1/s_1 \approx 1/(r_2/s_2)$, as expected from Eq. (141).

6.2.3 Rules for differentiation and change of variables in integrals

Given a composite function $h(x) = f(g(x))$ and its derivative

$$\frac{dh}{dx} = \frac{d}{dx}f(g) = \frac{df}{dg}\frac{dg}{dx} \tag{142}$$

via the chain rule. Now let F be the antiderivative of f, i.e.,

$$F(y) = \int f(y)dy \tag{143}$$

and so

$$F(y)' = \frac{dF}{dy} = f(y).$$

If it is the case that $y = g(x)$ then $F(g(x))$ is a composite function and by the chain rule,

$$\frac{d\,F(g)}{dx} = F'\frac{dg}{dx} = f\frac{dg}{dx}.$$

Upon integrating the left and right sides,

$$F(g(x)) = \int f(g)\frac{dg}{dx}dx. \tag{144}$$

If F is the antiderivative of $\int f(y)dy$, then $F(g)$ is the antiderivative of $\int f(g(x))\frac{dg}{dx}dx$.

We can also transform the independent variable of an integral by direct substitution of $y = g(x)$, and

so

$$\int_{y_1}^{y_2} f(y)dy = \int_{x_1=g^I(y_1)}^{x_2=g^I(y_2)} f(g(x))\frac{dg}{dx}dx. \tag{145}$$

The most interesting part of this is $dy \rightarrow \frac{dg}{dx}dx$, which accounts for the stretching or compression of the differential length. (In a two- or three-dimensional geometry this will be the determinant of the Jacobian of the transformation, examples below.) Thus we can change the independent variable y of the integral of Eq. (143) to x in Eq. (145), provided that $y = g(x)$ and x can be thought of as the inner function and independent variable of a composite function. This result will be invoked in Sec. 3.3 to change an integral over $\tilde{x}$, which is time-dependent when tracking a moving material volume, to the equivalent integral over x, the field coordinate and which is time-independent (the Reynolds Transport Theorem).

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Resource: Topics in Fluid Dynamics

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